PDHonline Course H146 (4 PDH)

Hydraulic Engineering

Instructor: Mohamed Elsanabary, Ph.D., PEng

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5272 Meadow Estates Drive
Fairfax, VA 22030-6658
Phone: 703-988-0088
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Hydraulic Engineering

Session 1

By

Dr. Mohamed Helmy Elsanabary

Session Goals

- Understand dimensions, units, and dimensional homogeneity
- Understand benefits of dimensional analysis
- Know how to use the method of repeating variables (π Theorem)
- Understand the concept of similarity and how to apply it to experimental modeling (Model Studies and Similitude).

Dimensions and Units

Primary Dimensions:

Quantity	Symbol	Dimensions
Length	L	L
Area	A	L^2
Volume	$\forall$	L^3
Velocity	U	LT^{-1}
Speed of sound	a	LT^{-1}
Volume flow rate	Q	L^3T^{-1}
Mass flow rate	m	MT^{-1}
Pressure, stress	p, σ	$ML^{-1}T^{-2}$
Strain rate	e	T^{-1}
Angle	θ	None
Angular velocity	ω	T^{-1}
Dynamic viscosity	μ	$ML^{-1}T^{-1}$
Kinematic viscosity	ν	L^2T^{-1}
Surface tension	σ	MT^{-2}
Force	F	MLT^{-2}
Moment, torque	M	ML^2T^{-2}
Power	P	ML^2T^{-3}
Work, energy	W, E	ML^2T^{-2}
Density	ρ	ML^{-3}
Temperature	T	θ
Specific heat	c_p, c_v	$\theta^{-1}L^2T^{-2}$
Thermal conductivity	k	$\theta^{-1}MLT^3$
Coefficient of thermal expan-	β	θ^{-1}

Dimensional Homogeneity

- Law of dimensional homogeneity (DH): every additive term in an equation must have the same dimensions
- Example: Bernoulli equation

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

- {p} = {force/area}={mass x length/time x 1/length^2} = {m/(t^2L)}
- {1/2ρV^2} = {mass/length^3 x (length/time)^2} = {m/(t^2L)}
- {ρgz} = {mass/length^3 x length/time^2 x length} ={m/(t^2L)}

Non-dimensionalization of Equations

- Given the law of DH, if we divide each term in the equation by a collection of variables and constants that have the same dimensions, the equation is rendered non dimensional
- In the process of non-dimensionalizing an equation, non-dimensional parameters often
- appear, e.g., Reynolds number and Froude number

Non-dimensionalization of Equations

- To non-dimensionalize, for example, the Bernoulli equation, the first step is to list primary dimensions of all dimensional variables and constants

$$p + \frac{1}{2}\rho V^2 + \rho g z = C$$

$\{p\} = \{m/(t^2 L)\}$ $\{\rho\} = \{m/L^3\}$ $\{V\} = \{L/t\}$

$\{g\} = \{L/t^2\}$ $\{z\} = \{L\}$

- Next, we need to select <u>Scaling Parameters.</u> For this example, select L, U_0, ρ_0

Non-dimensionalization of Equations

- By inspection, non-dimensionalize all variables with scaling parameters

$$p^\star = \frac{p}{\rho_0 U_0^2} \qquad \rho^\star = \frac{\rho}{\rho_0} \qquad V^\star = \frac{V}{U_0}$$

$$g^\star = \frac{gL}{U_0^2} \qquad z^\star = \frac{z}{L}$$

Back-substitute *p*, ρ, *V*, *g*, *z* into dimensional equation

$$\rho_0 U_0^2 p^\star + \frac{1}{2}\rho_0 \rho^\star \left(U_0^2 V^{\star 2}\right) + \rho_0 \rho^\star g^\star U_0^2 z^\star = C$$

Non-dimensionalization of Equations

- Divide by $\rho_0 U_0^2$ and set $\rho^* = 1$ (incompressible flow)

$$p^\star + \frac{1}{2}V^{\star 2} + g^\star z^\star = \frac{C}{\rho_0 U_0^2} = C^\star$$

- Since $g^* = 1/Fr^2$, where

$$p^\star + \frac{1}{2}V^{\star 2} + \frac{1}{Fr^2} z^\star = C^\star$$

Buckingham π Theorem

- m number of independent variables or physical parameters, which contain n number of primary dimensions (L, M, T), can form π_i independent, non-dimensional groups
- L (length)
  M (mass)
  T (time)

Buckingham π Theorem

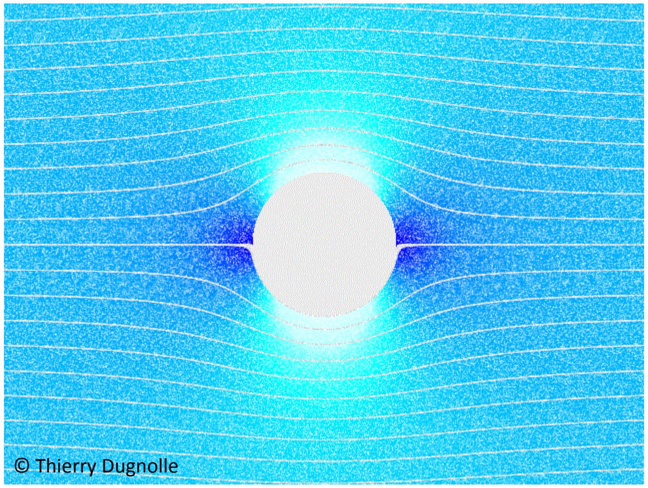
- In order to obtain the dimensionless groups πi, select some repeating variables r that contain some primary dimensions (L, M, T) but do not form dimensionless groups
- The π_i parameters are formed by combining each of the remaining m - r parameters with the repeating variables: π_i = m - r

Buckingham π Theorem

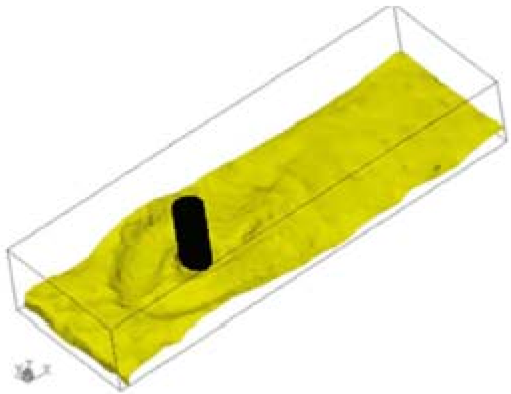
- For most problems r = n (# of primary dimensions)
- In some cases r < n
- Occurs when repeating parameters (r) are dependent on other variables (i.e., they are a combination of other repeating parameters)

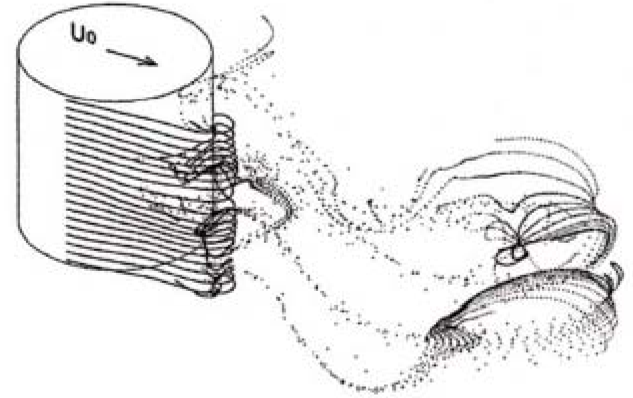
Flow Past Cylinder

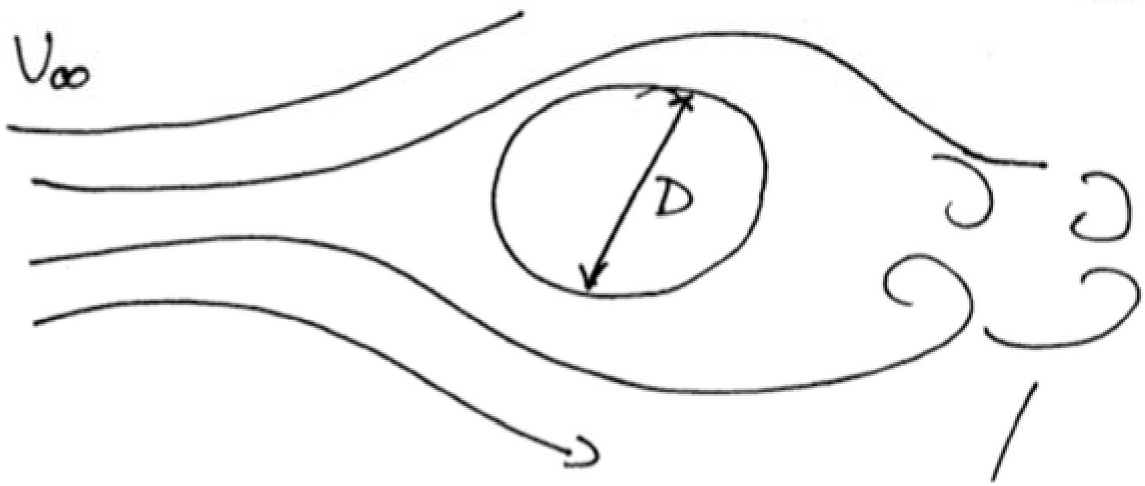
- Generally the unsteady force induced by the flow on the cylinder is a function of F, V_∞, μ, ρ, D, t :
- Notice that there are 6 independent variables, all in dimensional form
- By means of dimensional analysis (Buckingham π Theorem), the dependence between the variables is reduced to 3:



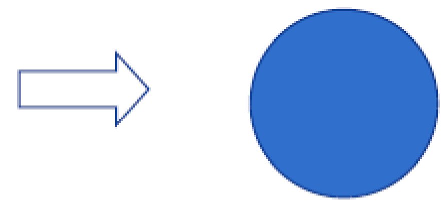
Flow Past Cylinder







Example



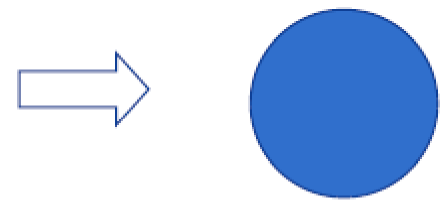
F_L = 0 because of symmetry

F_D = f(U,D,r,m)

To carry out a set of experiments to characterize the drag forces on spheres, one need to vary four parameters independently: U,D,r,m.

- It is very time consuming and the resulting data set will be difficult to analyze
- Some kinds of scaling are needed to apply the wind tunnel testing data to a real flow problem.

Example



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Example

- □ Step 1: Identify number of independent variables or physical parameters
  - ➢ F, V_∞, μ, ρ, D, t → m = 6
- Step 2: Identify number of primary dimensions
  - ➢ M, L, T → n = 3

Example

- Step 3: Identify number of repeating variables
  - $r = n = 3$
  - V_∞, ρ, D will appear in all π_i dimensionless groups
- Step 4: Identify number of π_i parameters
  - $\pi_i = m - r = 6 - 3 = 3$

Example

- The equation for the force F can be written as:
- And...

π_i (F, V∞, ρ, D) Repeating Variables, r

Example

- Step 5: Find dimensionless group
- Finding the powers in the equation:

$$M:\ 1+\gamma=0\rightarrow\gamma=-1$$
$$T:\ -2-\alpha=0\rightarrow\alpha=-2$$
$$L:\ 1+\alpha+\beta-3\gamma=0\rightarrow\gamma=-2$$

$$\pi_1=F\left[V_\infty\right]^{-2}\left[D\right]^{-2}\left[\rho\right]^{-1}=\frac{F}{\rho V_\infty^2 D^2}$$

Example

- Find π_2 $[\mu][V_\infty]^\alpha[D]^\beta[\rho]^\gamma = L^0T^0M^0$

$$\left[\frac{ML}{T}\right]\left[\frac{L}{T}\right]^\alpha [L]^\beta \left[\frac{M}{L^3}\right]^\gamma = L^0T^0M^0$$

$$M:\ 1+\gamma=0 \rightarrow \gamma=-1$$
$$T:\ -1-\alpha=0 \rightarrow \alpha=-1$$
$$L:\ -1+\alpha+\beta-3\gamma=0 \rightarrow \gamma=-1$$

$$\pi_2 = \mu[V_\infty]^{-1}[D]^{-1}[\rho]^{-1} = \frac{\mu}{\rho V_\infty D} = \frac{1}{\mathrm{Re}_D}$$

Example

- Find π_3

$$[t][V_\infty]^\alpha [D]^\beta [\rho]^\gamma = L^0 T^0 M^0$$

$$[T]\left[\frac{L}{T}\right]^\alpha [L]^\beta \left[\frac{M}{L^3}\right]^\gamma = L^0 T^0 M^0$$

$$M:\ \gamma = 0$$
$$T:\ 1-\alpha = 0 \rightarrow \alpha = 1$$
$$L:\ \alpha + \beta - 3\gamma = 0 \rightarrow \beta = -1$$

$$\pi_3 = t[V_\infty]^1 [D]^{-1} [\rho]^0 = \frac{tV_\infty}{D}$$

Example Answer

$$\boxed{\frac{F}{\rho V_\infty^2 D^2} = f\left(\frac{1}{\mathrm{Re}_D}, \frac{tV_\infty}{D}\right)}$$

Examples

Title	1000 solved problems in fluid mechanics (includes hydraulic machines) Sigma series
Author	K. Subramanya
Publisher	Tata McGraw-Hill, 2005
ISBN	0070583862, 9780070583863

Examples

6.8 A small sphere of density ρ_s and diameter D settles at a terminal velocity V in a liquid of density ρ_f and dynamic viscosity μ. Gravity g is known to be a parameter. Express the functional relationships between these variables in a dimensionless form.

Solution:

$$V = \text{fn}\,[\rho_s, D, \rho_f, \mu, g]$$

List the dimensions of each variable as follows:

V	ρ_s	D	ρ_f	μ	g
$[LT^{-1}]$	$[ML^{-3}]$	$[L]$	$[ML^{-3}]$	$[ML^{-1}\,T^{-1}]$	$[LT^{-2}]$

There are six variables, $n = 6$

and 3 basic dimensions, $m = 3$

Hence there are $(6 - 3) = 3$ dimensionless terms.

Select D, ρ_f and g as the repeating variables

I term: $\pi_1 = VD^a\rho_f^b g^c$

Following the usual procedure

$$[M^0L^0T^0] = [LT^{-1}]\,[L]^a\,[ML^{-3}]^b\,[LT^{-2}]^c$$

$$b = 0$$

$$1 + a - 3b + c = 0$$

$$-1 - 2c = 0$$

Hence $b = 0,\ c = -1/2,\ a = -1/2$

$$\therefore \quad \pi_1 = \frac{V}{\sqrt{gD}}$$

II term: $\pi_2 = \rho_s D^a \rho_f^b g^c$

By inspection, it is easy to see $\pi_2 = \dfrac{\rho_s}{\rho_f}$

III term: $\pi_3 = \mu D^a \rho_f^b g^c$

In this case, we have

$$[M^0L^0T^0] = [ML^{-1}\,T^{-1}]\,[L]^a\,[ML^{-3}]^b\,[LT^{-2}]^c$$

$$1 + b = c$$

$$-1 + a - 3b + c = 0$$

$$-1 - 2c = 0$$

$$b = -1,\ c = -\frac{1}{2} \text{ and } a = 3b - c + 1$$

$$= -\frac{3}{2}$$

$$\therefore \quad \pi_3 = \frac{\mu}{D^{3/2}\rho_f g^{1/2}}$$

Hence $$\frac{V}{\sqrt{gD}} = \varphi\left[\frac{\rho_s}{\rho_f}, \frac{\mu}{\rho_f D\sqrt{gD}}\right]$$

Remarks

- In practice for bluff bodies:

Drag Coefficient
$$C_D = \frac{F_D}{\frac{1}{2}\rho V_\infty^2 D^2}$$

Drag Force (F_D); Dynamic Pressure ($\frac{1}{2}\rho V_\infty^2$)

Lift Coefficient
$$C_L = \frac{F_L}{\frac{1}{2}\rho V_\infty^2 D^2}$$

Lift Force (F_L)

Remarks

- Exact dependence of F versus the ReD^{-1} and the dimensionless time is unknown; however now it is known that these parameters are key in the problem.
- ReD indicates the range of the flow (laminar, transitional, turbulent), so the behaviour of the dimensionless force will depend on this range.

Similarity

- **Geometric Similarity** - the model must be the same shape as the prototype. Each dimension must be scaled by the same factor
- **Kinematic Similarity** - velocity as any point in the model must be proportional
- **Dynamic Similarity** - all forces in the model flow scale by a constant factor to corresponding forces in the prototype flow
- **Complete Similarity** is achieved only if all 3 conditions are met. This is not always possible, e.g., river hydraulics models.

Model Studies and Similitude

TABLE 2-2. Common Dimensionless Groups in Fluid Mechanics.

Parameter	Definition	Qualitative ratio of effects	Importance
Reynolds number	$Re = \frac{UL}{\nu}$	$\frac{\text{inertia}}{\text{viscosity}}$	if friction due to kinetic viscosity is important
Froude number	$Fr = \frac{U}{\sqrt{gL}}$	$\frac{\text{inertia}}{\text{gravity}}$	free-surface flows if gravity is important
Mach number	$Ma = \frac{U}{a}$	$\frac{\text{inertia}}{\text{compressibility}}$	compressible flow
Weber number	$We = \frac{\rho U^2 L}{\sigma}$	$\frac{\text{inertia}}{\text{surface tension}}$	shallow free-surface flows
Euler number	$Eu = \frac{p - p_o}{\rho U^2}$	$\frac{\text{pressure}}{\text{inertia}}$	enclosed flows and free-surface flows
Cavitation number	$Ca = \frac{p - p_v}{\rho U^2}$	$\frac{\text{pressure}}{\text{inertia}}$	cavitation
Drag or lift coefficient	$C_D, C_L = \frac{F_D, F_L}{0.5\rho U^2}$	$\frac{\text{drag force, lift force}}{\text{dynamic force}}$	flow generally
Prandtl number	$Pr = \frac{\mu c_p}{k}$	$\frac{\text{dissipation}}{\text{conduction}}$	if molecular diffusion is important
Eckert number	$Ec = \frac{U^2}{c_p T_o}$	$\frac{\text{kinetic energy}}{\text{enthalpy}}$	dissipation of flow energy
Specific-heat ratio	$\frac{c_p}{c_v}$	$\frac{\text{enthalpy}}{\text{internal energy}}$	compressible flow
Strouhal number	$St = \frac{\omega L}{U}$	$\frac{\text{oscillation speed}}{\text{mean speed}}$	oscillating flow
Roughness ratio	$\frac{k}{L}$	$\frac{\text{wall roughness}}{\text{body length}}$	turbulent flow near rough boundaries
Grashof number	$Gr = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\frac{\text{buoyancy}}{\text{viscosity}}$	natural convection
Temperature ratio	$\frac{\Delta T_w}{\Delta T_o}$	$\frac{\Delta\ \text{wall temperature}}{\Delta\ \text{flow temperature}}$	heat transfer

Adopted from: Ettema (2000), Hydraulic Modeling: Concepts and Practice

Model Studies and Similitude

- Dynamic similitude
  - ➢ Geometric similitude
    - o All linear dimensions must be scaled identically
    - o Roughness must scale
  - ➢ kinematic similitude
    - o Constant ratio of dynamic pressures at corresponding points
    - o Streamlines must be geometrically similar
    - o Mach, Reynolds, Froude, and Weber numbers must be the same

Relaxed Similitude Requirements

- Impossible to have all force ratios the same unless the model is the Same Size as the prototype
- Need to determine which forces are important and attempt to keep those force ratios the same

Similitude Examples

- Open hydraulic structures
- Ship's resistance
- Closed conduit
- Hydraulic machinery

Similitude Examples

Examples

- spillways
- channel transitions
- Weirs

- **Important Forces**
  - Viscous forces (often small relative to gravity forces)
  - Gravity: from changes in water surface elevation
  - Inertial forces

$$Re = \frac{UL}{\nu}$$

$$Fr = \frac{U}{\sqrt{gL}}$$

- **Minimum similitude requirements**
  - Geometric
  - Froude number

Froude Similarity

- Froude number the same in model and prototype ($F_m=F_p$)

$$\frac{V_m^2}{g_m L_m} = \frac{V_p^2}{g_p L_p}$$

- Difficult to change g ($g_m=g_p$)
- Define length ratio (usually larger than 1)

$$L_r = \frac{L_p}{L_m}$$

- Velocity ratio

$$V_r = \sqrt{L_r}$$

- Time ratio

$$t_r = \frac{L_r}{V_r} = \sqrt{L_r}$$

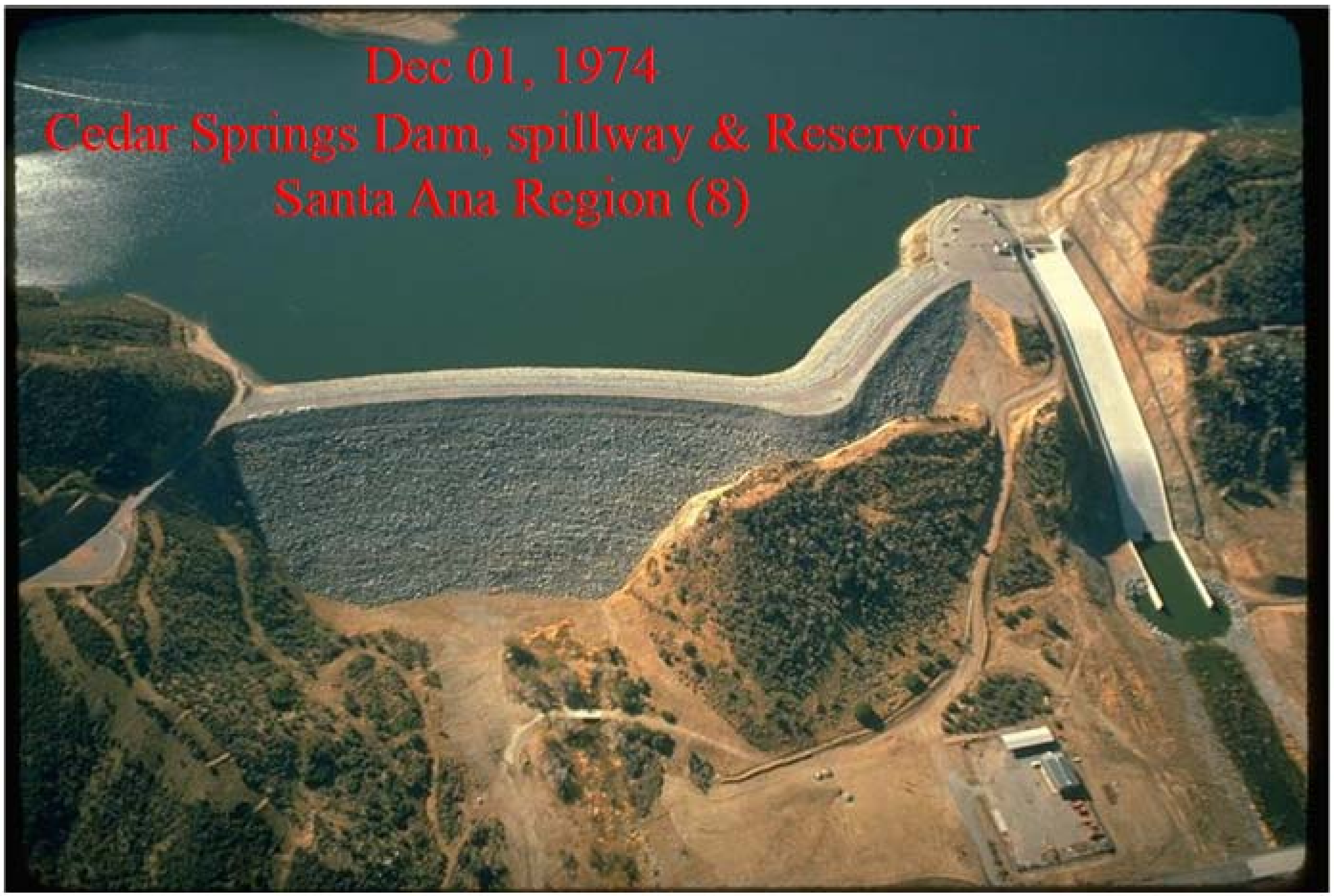
- Discharge ratio

$$Q_r = V_r A_r = \sqrt{L_r} L_r L_r = L_r^{5/2}$$

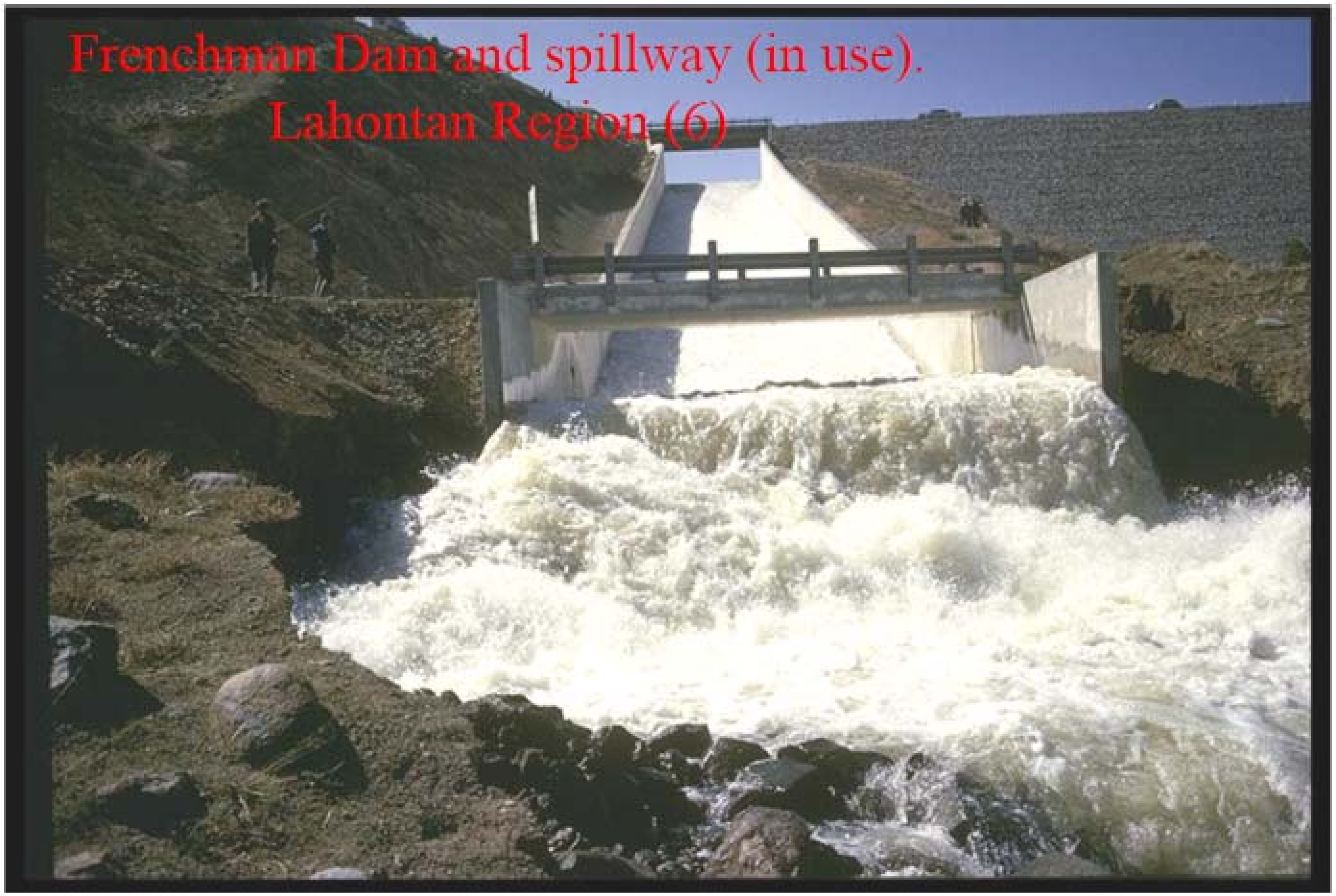
- Force ratio

$$F_r = M_r a_r = \rho_r L_r^3 \frac{L_r}{t_r^2} = L_r^3$$

Example: Spillway Model



Example: Spillway Model



Example: Spillway Model

A 50 cm tall scale model of a proposed 50 m spillway is used to predict prototype flow conditions. If the design flood discharge over the spillway is 20,000 m3/s, what water flow rate should be tested in the model?

$F_m = F_p$ $L_r = 100$

$Q_r = L_r^{5/2} = 100{,}000$

Therefore, $Q_m = \dfrac{20{,}000\, m^3/s}{100{,}000} = 0.2\, m^3/s$

Example

Similitude

6.11 A fluid flow phenomenon is to be studied in a model which is to be constructed by using Reynolds model law. Find the expressions for model to prototype ratios of velocity, discharge, pressure, work and power.

Solution: Using the subscripts m for model, p for prototype and r for the ratio of model to prototype:

In Reynolds law $Re = \left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p$

Let $L_r = \frac{L_m}{L_p}$, then

Velocity ratio $\frac{V_m}{V_p} = V_r = \frac{\mu_r}{\rho_r L_r} = \left(\frac{v_r}{L_r}\right)$

where v_r = ratio of the kinematic viscosities $= \frac{\mu_r}{\rho_r}$

Discharge ratio $\frac{Q_m}{Q_p} = Q_r = V_r L_r^2$

$$= \frac{\mu_r L_r}{\rho_r} = (v_r L_r)$$

Pressure ratio $\frac{p_m}{p_p} = p_r$

Work =F * L

Power = Work/Time

P = F*L/T

Example

6.16 Estimate for a 1/20 model of a spillway

(i) the prototype velocity corresponding to a model velocity of 1.5 m/s

(ii) the prototype discharge per unit width corresponding to a model discharge per unit width of 0.2 m^3/s per metre

(iii) the pressure head in the prototype corresponding to a model pressure head of 5 cm of mercury at a point

(iv) The energy dissipated per second in the model corresponding to a prototype value of 1 kW.

Solution: For dynamic similarity Froude number must be the same in the model and prototype. If L_r is the length ratio, then

(i) $$V_r = \frac{V_m}{V_p} = \sqrt{L_r}$$

$$V_p = V_m/\sqrt{L_r} = 1.5\sqrt{20} = \mathbf{6.71\ m/s}$$

(ii) ratio of discharge per unit width $= q_r = \dfrac{(Q/L)_m}{(Q/L)_p}$

$$q_r = \frac{Q_r}{L_r} = V_r L_r = L_r^{3/2}$$

$$q_p = q_m/L_r^{3/2} = 0.20 \times (20)^{3/2} = \mathbf{17.89\ m^3/s/m}$$

(iii) pressure ratio $p_r = (L_r \rho_r)$

Assume $\rho_m = \rho_p$, i.e. $\rho_r = 1.0$

Hence $p_r = L_r$

$\therefore$ $p_p = p_m/L_r = 5 \times 20$

$= \mathbf{100\ cm\ of\ mercury}$

(iv) Power ratio = (Energy loss/second)$_r$

$= [L_r^{7/2}\ \rho_r]$

As $\rho_r = 1.0$ (assumed), $\mathbf{P}_r = L_r^{7/2}$

$$\mathbf{P}_m = \mathbf{P}_p \cdot L_r^{7/2}$$

$$= 1000 \times \left(\frac{1}{20}\right)^{7/2} = \mathbf{0.028\ W}$$

6.17 Oil of density 917 kg/m^3 and dynamic viscosity 0.29 Pa.s flows in a pipe of diameter 15 cm at a velocity of 2.0 m/s. What would be the velocity of flow of water in a 1.0 cm diameter pipe, to make the two flows dynamically similar? The density and viscosity of water can be taken as 998 kg/m^3 and 1.31×10^{-3} Pa.s respectively.

Solution:

Reynolds similarity law is applicable.

$$(Re)_m = \frac{V_m d_m}{\nu_m} = (Re)_p = \frac{V_p d_p}{\nu_p}$$

$$\therefore \quad V_r = \frac{V_m}{V_p} = \frac{\nu_r}{L_r} = \frac{\mu_r}{L_r \rho_r}$$

$$\frac{V_m}{V_p} = \frac{\mu_m}{\mu_p} \frac{1}{\left(\frac{L_m}{L_p}\right)\left(\frac{\rho_m}{\rho_p}\right)}$$

Referring to oil with a subscript p and water with a suffix m

$$\frac{V_m}{V_p} = \frac{1.31 \times 10^{-3}}{0.29} \times \frac{1}{\left(\frac{1.0}{15.0}\right)\left(\frac{998}{917}\right)}$$

$= 0.0623$

V_m = Velocity of water flow $= V_p \times 0.0623$

$= 2 \times 0.0623 = \mathbf{0.1246\ m/s}$

Example

6.18 A 1 : 6 scale model of a passenger car is tested in a wind tunnel. The prototype velocity is 60 km/h. If the model drag is 250 N what is the drag and the power required to overcome the drag in the prototype? The air in the model and prototype can be assumed to have the same properties.

Solution: Reynolds similarity law is applicable.

$$(Re)_m = \frac{V_m L_m}{\nu_m} = (Re)_p = \frac{V_p L_p}{\nu_p}$$

$\therefore$ $V_r = \dfrac{\nu_r}{L_r}$

If $\nu_r = 1$ (i.e. $\rho_m = \rho_p$, $\mu_m = \mu_p$)

then $V_r = 1/L_r$.

$V_m = V_p/L_r = 60 \times 6 = 360$ km/h

$= 100$ m/s

Force ratio $\dfrac{F_m}{F_p} = \dfrac{\mu_r^2}{\rho_r}$

If $\rho_r = 1$, and $\mu_r = 1$, $\dfrac{F_m}{F_p} = 1.0$

$\therefore$ $\mathbf{F_p = 250\ N}$ (same as in the model)

Power to overcome drag in the prototype:

$$\mathbf{P}_p = F_p \cdot V_p = 250 \times \left(\frac{60 \times 10^3}{3600}\right) = 4167 \text{ W}$$

$= \mathbf{4.167\ kW}$

Example

6.19 The pressure drop in a flow meter in which an oil flows at an upstream velocity of 0.9 m/s is to be estimated by model studies. A 1 : 6 scale model using water is used. If the pressure drop in the model is 450 Pa, what will be the prototype pressure drop? If the prototype discharge is 200 L/s what is the model discharge? The following data are relevant:

Item	*Prototype*	*Model*
Density	900 kg/m^3	998 kg/m^3
Viscosity	0.104 Pa.s	1×10^{-3} Pa.s

Solution: The Reynolds model law is applicable.

$$(Re)_m = \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

$$L_m/L_p = L_r = 1/6$$

$$V_r = \frac{\mu_r}{L_r \rho_r}$$

Pressure ratio $\quad p_r = \rho_r V_r^2 = \dfrac{\mu_r^2}{L_r^2 \rho_r}$

and discharge ratio $\quad Q_r = V_r \cdot L_r^2 = \dfrac{\mu_r L_r}{\rho_r}$

In the present problem:

$$\mu_r = \frac{1\times10^{-3}}{0.104} = 9.615 \times 10^{-3}$$

$$\rho_r = \frac{998}{900} = 1.109$$

$$L_r = 1/6$$

$$p_r = \frac{(9.615\times10^{-3})^2}{\left(\frac{1}{6}\right)^2 \times 1.109} = 3 \times 10^{-3}$$

$$p_p = \frac{p_m}{p_r} = \frac{450}{3\times10^{-3}} = 150{,}000 \text{ Pa}$$

$$= \mathbf{150\ kPa}$$

$$Q_r = \frac{(9.615\times10^{-3})\times 1/6}{1.109} = 1.445 \times 10^{-3}$$

$$Q_m = Q_p \times 1.445 \times 10^{-3}$$

$$= 200 \times 1.445 \times 10^{-3} = \mathbf{0.289\ L/s}$$

Example

6.20 An underwater device is 1.5 m long, and is to move at 3.5 m/s. A geometrically similar model 30 cm long is tested in a variable pressure wind tunnel at a speed of 35 m/s. Calculate the pressure of air in the model. If the model exhibits a drag force 40 N, calculate the prototype drag force.
[Assume $\rho_{water} = 998$ kg/m^3, ρ_{air} at standard atmospheric pressure = 1.17 kg/m^3, $\mu_{air} = 1.90 \times 10^{-5}$ Pa.s, at local atmospheric pressure and $\mu_{water} = 1.0 \times 10^{-3}$ Pa.s].

Solution: Reynolds model law is applicable. Hence

$$(Re)_m = \frac{\rho_m V_m L_m}{\mu_m} = (Re)_p = \frac{\rho_p V_p L_p}{\mu_p}$$

$\therefore$ If $\frac{L_m}{L_p} = L_r$, $\frac{\rho_m}{\rho_p} = \rho_r$ and $\frac{\mu_m}{\mu_p} = \mu_r$

$$\frac{V_m}{V_p} = V_r = \frac{\mu_r}{\rho_r L_r}$$

In the present case

$$L_r = \frac{0.30}{1.50} = \frac{1}{5}$$

$$V_r = \frac{35}{3.5} = 10$$

$$\mu_r = \frac{1.90 \times 10^5}{1 \times 10^{-3}} = 1.9 \times 10^{-2}$$

Hence $\rho_r = \frac{\mu_r}{V_r L_r} = \frac{1.9 \times 10^{-2}}{10 \times 1/5} = 0.0095$

Hence $\rho_{air} = \rho_m = 998 \times 0.0095 = \mathbf{9.481}$

This is about 8 times larger than the density at atmospheric pressure. Since at constant temperature, by the equation of state p/ρ = constant.

Hence $\frac{(\text{pressure})_{model}}{(\text{Atmospheric pressure})} = \frac{\rho_{model}}{\rho_{atmospheric}}$

$$p_{model} = \frac{9.481}{1.17} \times p_a = 8.103\ p_a$$

= 8.103 times local atmospheric pressure

Force ratio $= F_r = \rho_r V_r^2 L_r^2$

$$= \frac{\mu_r^2}{\rho_r} = \frac{(1.9 \times 10^{-2})^2}{0.0095}$$

= 0.038

$$F_p = \frac{F_m}{0.038} = \frac{40}{0.038} = 1053 \text{ N}$$

= **1.053 kN**

Example

6.21 A pipe of diameter 1.5 m is required to transport an oil of relative density 0.9 and kinematic viscosity = 3×10^{-2} stoke at a rate of 3.0 m^3/s. If a 15 cm diameter pipe with water at 20°C ($\nu = 0.01$ stoke) is used to model the above flow, find the velocity and discharge in the model.

Solution: The Reynolds number must be the same in the model and prototype for similar pipe flows.

$$\frac{V_p D_p}{\nu_p} = \frac{V_m D_m}{\nu_m}$$

$$V_m = V_p \frac{D_p}{D_m} \frac{\nu_m}{\nu_p}$$

$$V_p = \frac{Q}{\pi D^2/4} = \frac{3.0}{\frac{\pi}{4} \times (1.5)^2} = 1.6977 \text{ m/s}$$

$$V_m = 1.6977 \times \frac{1.5}{0.15} \times \frac{0.01}{0.03} = \mathbf{5.659 \text{ m/s}}$$

$$Q_m = \text{discharge in the model} = \frac{\pi}{4} D_m^2 \times V_m$$

$$= \frac{\pi}{4} \times (0.15)^2 \times (5.659) = \mathbf{0.1 \text{ m}^3/\text{s}}$$

Experimental Testing and Incomplete Similarity

1/20th scale model

- For ship hydrodynamics, Fr similarity is maintained while Re is allowed to be different.
- Why? Look at complete similarity:

$$Re_p = \frac{V_p L_p}{\nu_p} = Re_m = \frac{V_m L_m}{\nu_m} \rightarrow \frac{L_m}{L_p} = \frac{\nu_m}{\nu_p}\frac{V_p}{V_m}$$

$$Fr_p = \frac{V_p}{\sqrt{gL_p}} = Fr_m = \frac{V_m}{\sqrt{gL_m}} \rightarrow \frac{L_m}{L_p} = \left(\frac{V_p}{V_m}\right)^2$$

- To match both Re and Fr, viscosity in the model test is a function of scale ratio! This is not feasible.

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$

Ship's Resistance

- Skin friction (**Viscosity, Roughness**)
- Wave drag (free surface effect) (**Gravity**)
- Therefore we need (**Reynolds**) and (**Froude**) similarity

$$\frac{2\,Drag}{\rho V^2 A} = C_d = f\left(\frac{\varepsilon}{l}, R, F\right)$$

Reynolds and Froude Similarity?

Reynolds

$$R = \frac{\rho V l}{\mu}$$

$$\frac{\rho_m V_m l_m}{\mu_m} = \frac{\rho_p V_p l_p}{\mu_p}$$

$$V_m l_m = V_p l_p$$

$$\frac{V_p}{V_m} = \frac{l_m}{l_p}$$

$$V_r = \frac{1}{L_r}$$

Water is the only practical fluid

Froude

$$F = \frac{V}{\sqrt{gl}}$$

$$V_r = \sqrt{L_r}$$

$$\frac{1}{L_r} = \sqrt{L_r}$$

$$L_r = 1$$