PDHonline Course S316 (6 PDH)

Parabolic Arch Bridges

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1.0 INTRODUCTION

The arch bridge, constructed in various forms for over three thousand years, supports vertical loads through a curved surface to the foundations (abutments). This support is primarily axial, rather than flexural, in nature. Horizontal beams, on the other hand, support vertical loads by flexure.

Consider, for example, a span of forty (40) feet, with a uniform vertical load of one (1) kip/ft. Contrast a horizontal beam with a three-hinged parabolic arch of four (4) foot rise.

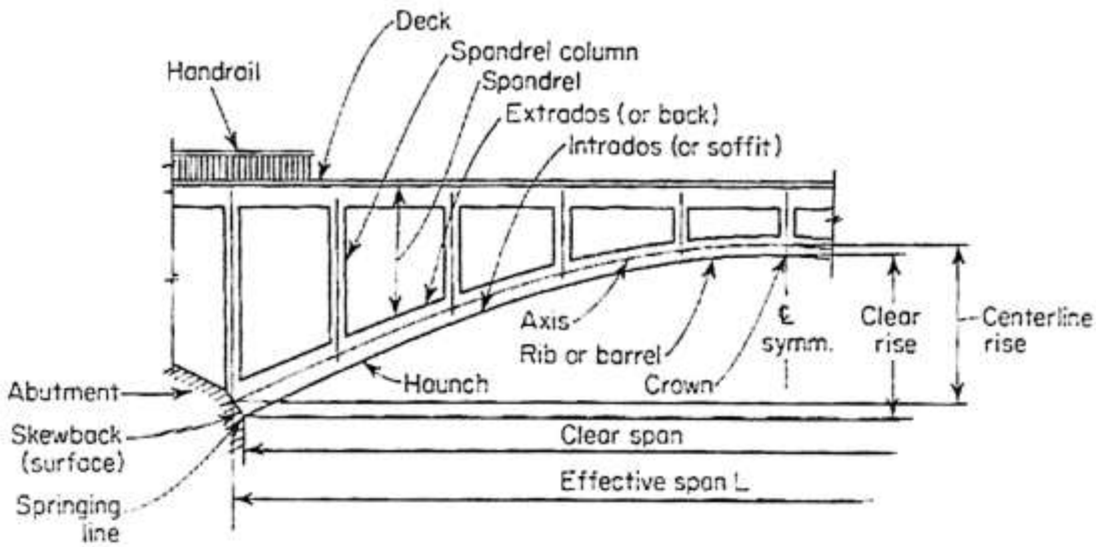
Load	Horizontal Beam	Parabolic Arch
Flexure	200 kip-ft	0
Axial	0	50 kip (crown)
Base horizontal	0	50 kip

This shows both the advantage of the arch in flexure, but also its disadvantage in base horizontal force (thrust).
Assuming the allowable flexural stress in a steel beam is 20 ksi and the allowable compressive stress is 5 ksi, then the beam requires a W14x82 while the arch a W14x24.
This advantage in flexure is strong for a constant uniform load, which holds in bridges because of the very large dead load of the roadway (or trainway) deck. In buildings, however, dead load is a relatively small load compared with others. The arch still has advantage in terms of less moment, but not as striking as that above.

The types of parabolic arches considered here are:

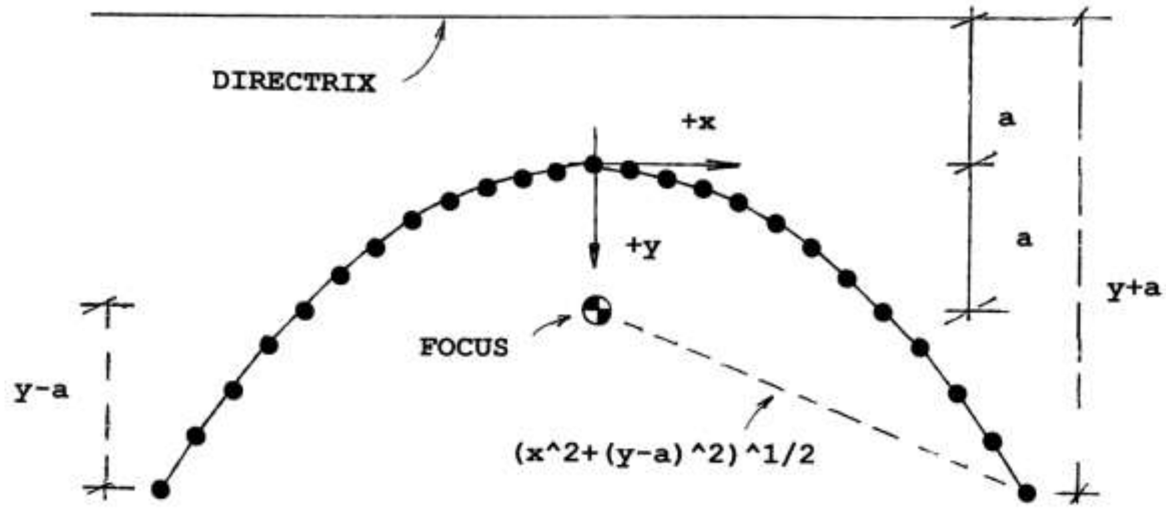
- 3 Hinged Arch – 2 hinges at abutments, 1 at crown
- 2 Hinged Arch - 2 hinges at abutments
- Fixed Arch - all connections fixed
- Tied Arch - opposite abutments structurally tied
- Deck Arch - roadway above arch
- Through Arch - portion, or all, of arch above roadway

Nomenclature on the following page is taken from Reference 1.



NOMENCLATURE

Arch	-	**consists of several parallel ribs or a single curved sheet (barrel)**
Soffit	-	**underside of arch**
Crown	-	**highest point of soffit**
Back	-	**top surface of arch**
Springline	-	**intersection of pier (or abutment) with soffit**
Haunch	-	**midsection of ribs**
Intrados	-	**intersection of soffit with vertical plane through crown and springline**
Extrados	-	**intersection of back with vertical plane through crown and springline**
Clear Rise	-	**height of intrados at crown above springline**
Spandrel	-	**span beween back of arch and roadway, may be open with vertical ribs or closed and filled with earth**



A parabola is the locus of points equidistant from a line, the directrix, and a point, the focus. Let this distance be 'a' as shown above.

From this definition, a+y = (x2^2+(y-a)^2)^1/2.
This equation can be solved to yield :
4*a*y = x^2 → y = x^2/4*a.

Given the rise, say f, and the span, say L, the quantity 'a' may be found as:
y = x^2/4*a → rise = (span/2)^2/4*a
Reconfiguring this formula,
a = span^2/16*rise

Example : What is the value of a for a rise of 2 feet, and a span of 40 feet?

Solution : a = 40^2/16*2 = 50 feet

Solving the formula above for span,
span = 16*a*rise
If rise = a, span = 4*a, a relationship that holds for all parabolas.

2.0 THREE-HINGED PARABOLIC ARCHES

Our study here covers symmetric three-hinged arches with abutments at the same level. These arches have hinges at both springing and at the crown, and are statically determinant.

Two load cases are shown on the next page, namely an isolated vertical point load on the right half of the arc, and a uniform vertical load.

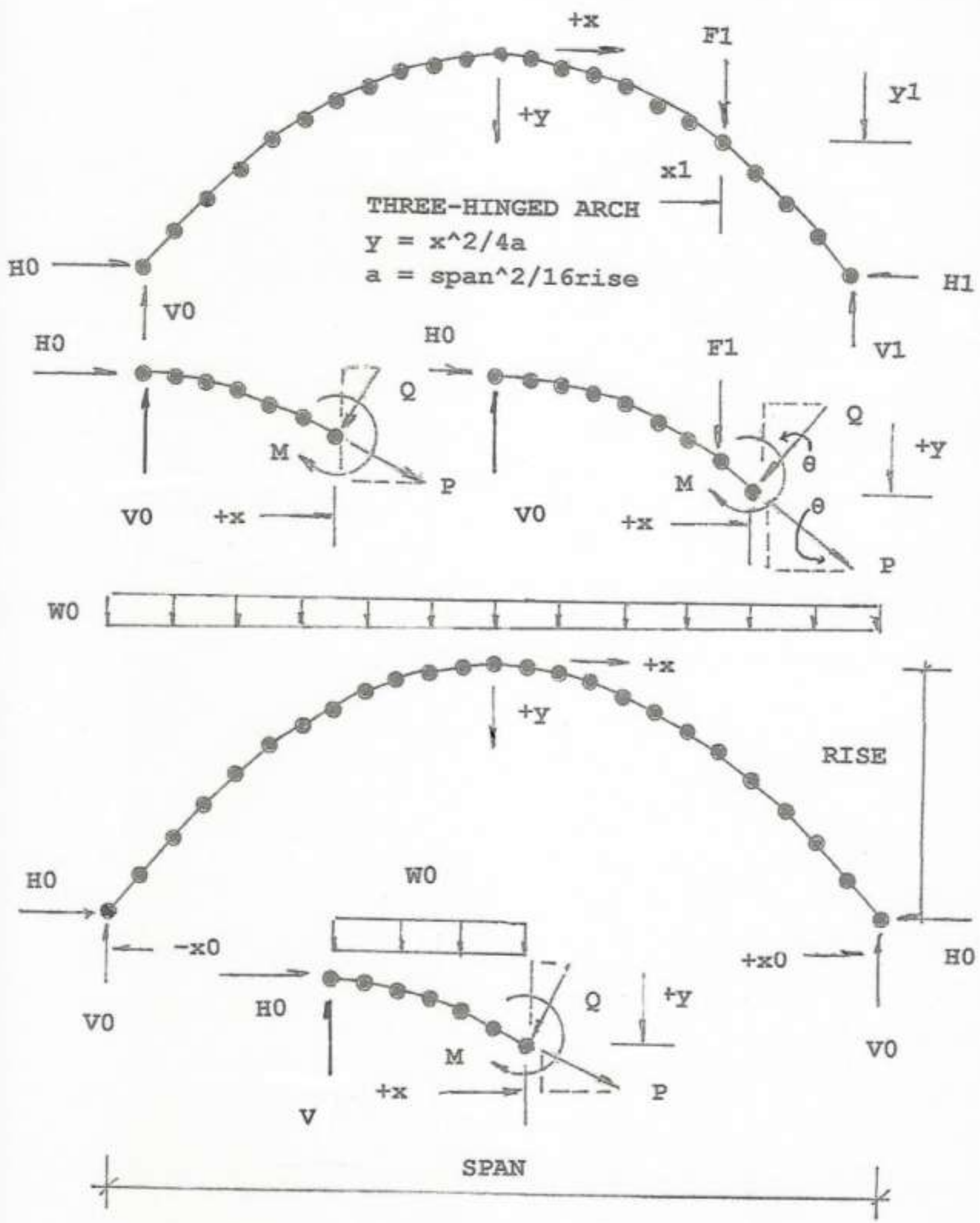
The diagram of the first load case shows two free body diagrams. The one on the left represents a segment of the arch from the crown to the point of application of the vertical load. The one on the right shows a free body segment from the crown to the right of the vertical load.

In all cases, P represents axial force, Q transverse force, and M moment with directions as shown. Θ represent the angle from the horizontal (+x axis) to P, as well as the angle from the vertical (+y axis) to Q. Note that this angle changes as x changes, resulting in more complicated calculations compared to a straight beam.

In all the following developments,
x0 = span/2
y0 = rise (positive downward)
M = positive clockwise

For the first case, consider the support reactions,
Sum of vertical forces = -V0-V1+F1 = 0
Sum of moments about right support = +V0*(2*x0)-F1*(x0-x1) = 0
Thus V0 = F1*(x0-x1)/2*x0
and V1 = F1*(x0+x1)/2*x0

Now take the moments about the right half of the arch to give:
H0*y0+V0*x0-F1*(x0-x1) = 0
This solves as H0 = F1*(x0-x1)/2*y0
Since there are not applied horizontal forces, H1 = H0,
and all the support reactions are known.


+x
F1
y1
+y
x1
THREE-HINGED ARCH
y = x^2/4a
a = span^2/16rise
H0
V0
H1
V1
H0
Q
M
P
V0
+x
H0
F1
Q
θ
+y
M
θ
V0
+x
P
W0
+x
+y
RISE
H0
W0
-x0
+x0
H0
V0
H0
Q
+y
V0
M
+x
P
V
SPAN

Knowing the support reactions, the equations of statics may be used to find the forces N axial and Q shear) and moment M on any point on the arch

Examining the free body diagram to the left of the vertical load we have:

$$+H0 + P*\cos\theta - Q*\sin\theta = 0$$
$$-V0 + P*\sin\theta + Q*\cos\theta = 0$$
$$+M + V0*x + H0*y = 0$$

where $\theta = \tan^{(-1)}(x/2*a)$ **and** $a = x0^2/4*y0$

From the free body diagram to the right of the vertical load we have:

$$+H0 + P*\cos\theta - Q*\sin\theta = 0$$
$$-V0 + P*\sin\theta + Q*\cos\theta + F1 = 0$$
$$+M + V0*x + H0*y - F1*(x-x1) = 0$$

In each case we have three equations in three unknowns.
They are solved as:
To the left of F1,

$$P = -H0*\cos\theta + V0*\sin\theta$$
$$Q = +H0*\sin\theta + V0*\cos\theta$$
$$M = -H0*y - V0*x$$

To the right of F1,

$$P = -H0*\cos\theta +(V0-F1)*\sin\theta$$
$$Q = +H0*\sin\theta +(V0-F1)*\cos\theta$$
$$M = +F1*(x-x1)-H0*y-V0*x$$

The equations for a point load to the left of the crown may be solved in a similar manner.

Now consider a uniform load as shown as the lower arch on the previous diagram sheet.
For the total arch, by uniform load and symmetry,

$$V0 = V1 = w0*L/2 = w0*x0$$

From a free body diagram of the left half,
Shear at center $= w0*L/2 - w0*L/2 = 0$
Thus V in at the crown = 0.

In the right half, take moments about the crown,

$$-H0*y0 + V0*x0 - w0*x0^2/2 = 0$$

Thus $H0 = w0*x0^2/2*y0$

For the right segment shown,
(1) H0 + P*cosθ −Q*sinθ = 0
(2) w0*x + P*sinθ +Q*cosθ = 0
(3) w0*x^2/2 + H0*y + M = 0
It can be shown that y = y)*x^2/x0 so that H0*y = w0*x^2/x0.
Thus M = 0 at each point.

Solving (1) and (2) gives:
P = -H0*cosθ – w0*x*sinθ
Q = +H0*sinθ – w0*x*cosθ
For the arch with span = 40′ and rise = 4′,
H0 = 50 kip
P base = 53.182 kip
Qcrown = Q hase = 0

The most famous three-hinged bridge is the one crossing the Salinga river ravine (Salingatobel in German) in Schiers, Switzerland. It was designed and constructed by Robert Maillart. He won the contract over eighteen other entries, both for a combination of price and appearance. The arch has a rise of 42.6 feet and a span of 295 feet. While not as long as some other arch bridges, the span may be visualized as the length of a football field. It rises at a 3% gradient, which is not considered in the following analysis. The bridge width of 12.46 feet serves pedestrian, auto, and truck traffic.

The wooden framework was constructed in 1929 by six men, as described in Reference 2. One of the scaffolders fell 115 feet and survived! This set back the form work which was not completed until late October, too late for concrete construction. The concrete construction was then accomplished in three months in 1930.

Robert Maillart developed the three-hinged, hollow-box arch and deck-stiffened arch bridge construction system, revolutionary concepts. He designed and constructed some twenty arch bridges.

The arch walls join the deck at the quarter points. This was probably due to two reasons. This provides a much stronger member flexural member at the high stress areas for non-uniform loads, and is much easier to construct formwork than closely spaced arch rib and traffic deck.

The next page shows the two views of the bridge (Reference 3) and the crown and springing hinges (Reference 4). The use of these three hinges allows the supports to move (a reasonable amount), and provides resistance against temperature changes.

Reference 4 describes the Salingatobel bridge as:

Span = 295 ft Rise = 42.6 ft

Uniform dead load = 5.7 kip/ft

Concentrated (vehicle) load) = 55 kip

The section at the ¼ points is a concrete rectangle with the following parameters calculated from the paper data:

A = 4293 in.^2

S = 202477 in.^3

Find: (1) Thrust at spring, ¼ points, and crown for dead load

(2) Thrust, shear, and moment at same points for 55 k Load at right ¼ point

(3) Tensile and compressive stresses at right ¼ point

Solution

(1)

location	P(kip)
Left springing	-1681
Left ¼ point	-1515
Crown	-1456
Right ¼ point	-1515
Right springing	-1681

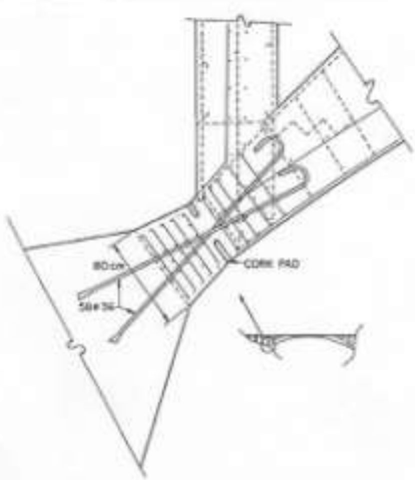
(2)

location	P(kip)	Q(kip)	M(kip-ft)
Left springing	-48	-12	0
Left ¼ point	-50	0	+507
Crown	-48	+14	0
Right ¼ point	-57	-26	-1521
Right springing	-62	-12	0

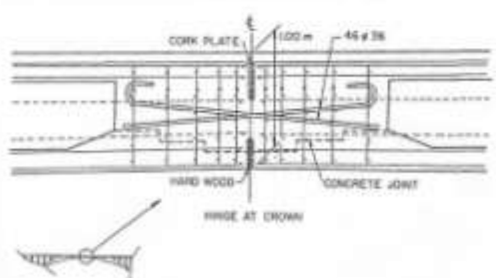
(3) axial stress = 353 psi (compression)

flexural stress = 90.1 psi

thus rib is under compression.



Hinge at springing line, Salginatobel bridge



Hinge at crown, Salginatobel bridge

3. TWO-HINGED AND FIXED ARCHES

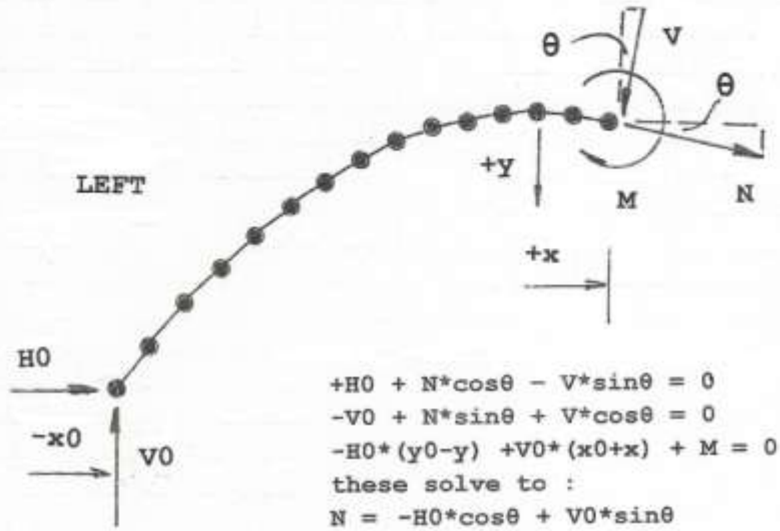
A method commonly used for linear elastic statically indeterminate plane structures is the "force method". The method obtains expressions for redundant supports, for portal frames, gable frames, and straight beams. Obtaining these redundant reduces the structure to one that is amenable to analysis as a statically determinant structure. Its drawback is the difficulty of using it for curved members.

A better method for obtaining these redundant is the method of "least work", published by Carlo Castigliano in 1869. This method has been used for over 125 years, and is applied by the author H. M. Martin to indeterminant trusses, continuous straight beams, curved beams, as well as three-hinged, two-hinged, and fixed arches, a formidable undertaking! At the time of publication of this book there was no electronic computational assistance. See Reference 5 for his work.

The basic principle of the method of least work is that the integral of the product of the strain energy with the partial derivative of the strain energy over the arch is zero. Strain energy may be thought of as potential energy stored in the member by flexural, axial, and shear displacements.
The method is complicated, by not impossible, to apply as shown by H.M. Martin. Here we assume the effects of flexural strain energy only.
Flexural strain energy = U = ∫(M^2/2EI)ds and the partial derivative of the strain energy with respect to H0 is dU/dH0.

The equation that must be solved is ∫M(dM/dH0)ds = 0 where the integration is taken over the entire arch.

The next page shows diagrams and equations to the left and right of a concentrated load, where H0 is the horizontal redundant force.



$+H0 + N*\cos\theta - V*\sin\theta = 0$

$-V0 + N*\sin\theta + V*\cos\theta = 0$

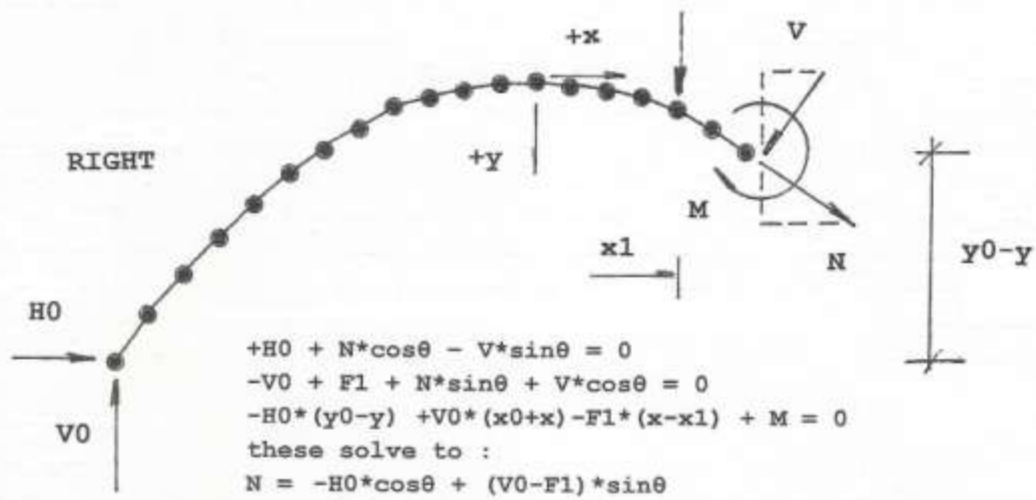
$-H0*(y0-y) + V0*(x0+x) + M = 0$

these solve to :

$N = -H0*\cos\theta + V0*\sin\theta$

$V = +V0*\cos\theta + H0*\sin\theta$

$M = -H0*(y-y0) - V0*(x+x0)$



$+H0 + N*\cos\theta - V*\sin\theta = 0$

$-V0 + F1 + N*\sin\theta + V*\cos\theta = 0$

$-H0*(y0-y) + V0*(x0+x) - F1*(x-x1) + M = 0$

these solve to :

$N = -H0*\cos\theta + (V0-F1)*\sin\theta$

$V = +(V0-F1)*\cos\theta + H0*\sin\theta$

$M = -H0*(y-y0) - V0*(x+x0) + F1*(x-x1)$

TWO-HINGED ARCH
CONCENTRATED LOAD

In this equation, E and I are constant, and the integration is done over the entire arch. Noting that dM/dH0 = -(y-y0), the equation above becomes

+(1/EI)∫(+H0(y-y0)^2+V0(x+x0)(y-y0)ds
+(1/EI)∫(+H0(y-y0)^2+V0(x+x0)(y-y0)-F1(x-x1)(y-y0))ds = 0

The first integration is done from x = -x0 to x1, and the second from x1 to +x0.
ds = (1+(dy/dx)^2)^(1/2) = (1+x^2/4a^2)^(1/2)

The solution to this problem, together with solutions incorporating axial and shear strain energies, is shown in on H0.c in Appendix I, for the interested reader.
To check these calculations, a second independent source is required. Chapter 9 of Reference 6 contains equations for two-hinged arches for vertical loads, horizontal loads, impressed distortions, and temperature effects.

For example, given an arch with:
span = 40 ft, rise = 2 ft, A = 6 in.^2, I = 18 in.^4
Consider two cases, a 10 kip load at the crown and a 10 kip load at the left quarter point.

	Leontovich			H0.c – flexure only		
	Mmax	Nmax	Vmax	Mmax	Nmax	Vmax
1st	262.500	39.666	5	262.925	39.349	5
2nd	399.023	28.762	5.257	398.926	28.768	5.258

	H0.c – flexure + axial			H0.c-flexure+axial+shear		
	Mmax	Nmax	Vmax	Mmax	Nmax	Vmax
1st	272.018	38.972	5	272.030	38.971	5
2nd	403.791	28.503	5.231	403.787	28.503	5.231

Note that Reference 6 is close to H0.c, flexure only. Note also the increase in moment and decrease in axial force when axial strain energy is included. It is also seen that, for this shallow arch, little precision is gained by including the shear strain energy.

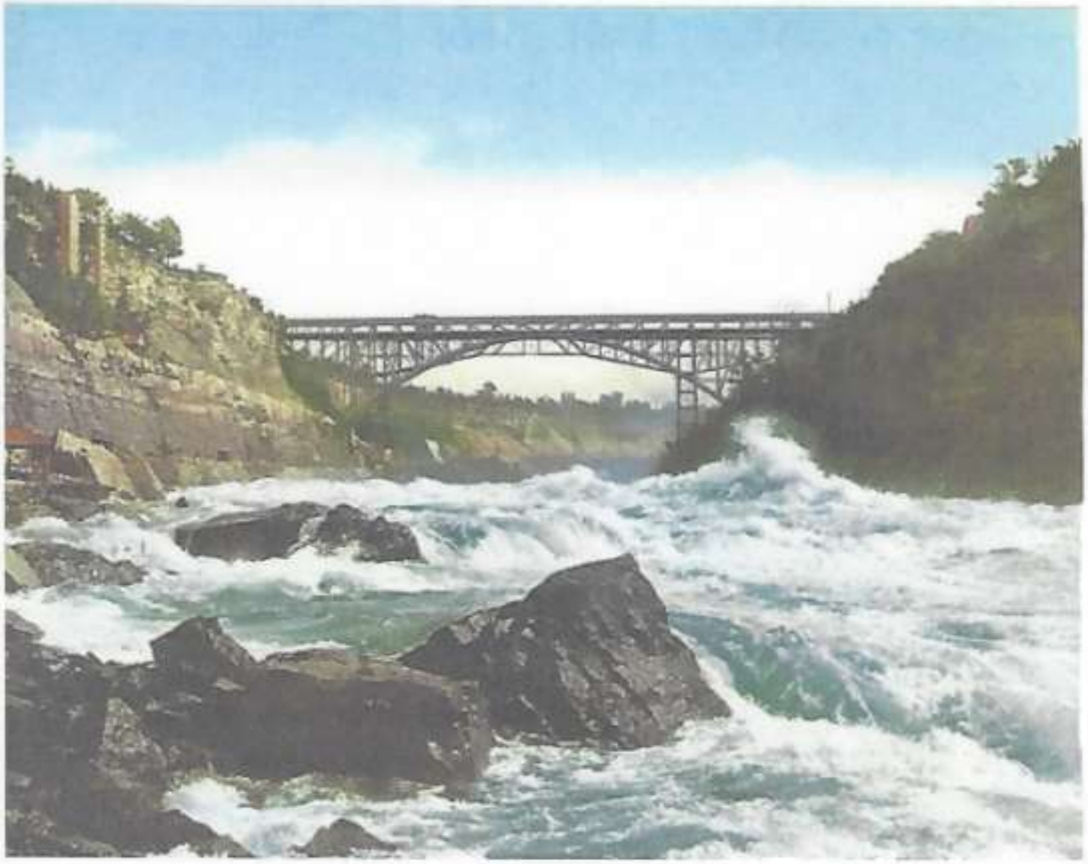
An example of two-hinge arches is the whirlpool rapids bridge in Niagara Falls, joining Canada and the United States. It was opened for traffic in 1897, and is still in use.

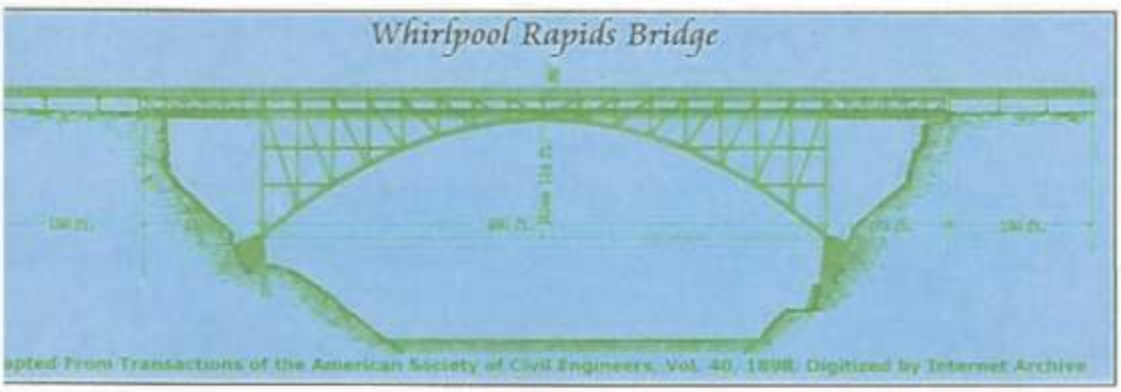
It is a steel deck arch bridge, with span of 550 feet, roadway width of 26 feet, and a rise of 114 feet. It is spandrel braced, riveted, and has two decks. The upper deck has a single lane for railway traffic and the lower deck two lanes for passenger vehicles only, no trucks or bicycles.

The photograph on the next page is from Reference 7 and the drawing from Reference 8.

For a fixed arch, using the method of least work, there are three (3) equations of strain energy for each redundant, i.e., M0, V0, and H0. This requires a very considerable mathematical effort, and will not be attempted here.

A major disadvantage of the least work method is that it does not lend itself to an analysis including the roadway and support columns and/or suspenders. These disadvantages will be overcome in the next two sections, again based on the work of Carlo Castigliano.




Whirlpool Rapids Bridge
apted From Transactions of the American Society of Civil Engineers, Vol. 40, 1898. Digitized by Internet Archive

4. PARABOLIC ARC FINITE ELEMENT

The use of a finite element for parabolic arches has several advantages over the classical methods. It is considerably easier to use,, may be used in combination with struts and/or hangers, and roadway for true arches, through arches, and tied arches.

The following development is similar to that in Reference 9, but the required integrations, coordinate transformations, and assembly with straight beam element below are the author's.

```
┌    ┐   ┌                             ┐ ┌    ┐
| N1 |   | k11 k12 k13 k14 k15 k16 | | u1 |
| V1 |   | k21 k22 k23 k24 k25 k26 | | v1 |
| M1 | = | k31 k32 k33 k34 k35 k36 | | ζ1 |
| N2 |   | k41 k42 k43 k44 k45 k46 | | u2 |
| V2 |   | k51 k52 k53 k54 k55 k56 | | v2 |
| M2 |   | k61 k62 k63 k64 k65 k66 | | ζ2 |
└    ┘   └                             ┘ └    ┘
```

In this model, the term kij relates the force (or moment) at degree of freedom (d.o.f.) i to the displacement (or rotation) at d.o.f. j . There are six (6) degrees of freedom in this planar element. For example, the term k26 relates the shear displacement at node 1 (V1) to the rotation node 2 (ζ2) as V1 = k26*ζ2.

The term "stiffness matrix" is coined because the larger the matrix term, the more force is required to maintain a given displacement, i.e., measure of stiffness.
The matrix is divided into four parts, each of dimension 3x3.

```
┌   ┐   ┌           ┐
|   |   | K11 | K12 |
| K | = |-----------|
|   |   | K21 | K22 |
└   ┘   └           ┘
```

K11 and K22 are each independent of the other 3x3 sub-matrices, while K21 is derived from K11 and K12 is the transpose of K21.

K11 and K22 are not found directly. Instead the corresponding flexibility matrices are found. The determination of these

matrices is the difficult part of the development, as will be seen. The remaining steps to obtain the 6x6 matrix use standard matrix algebra techniques.

In the following, only the salient items are discussed in the text. The main function of the computer program, arc5.c, is shown in Appendix II for the interested reader.

The following page shows a sketch of relevant items for an arch finite element with a x<0 and that for an element with x>0. The elements are shown as two (2) pieces each, for clarity. Here n = 1 or 2 for node 1 or node 2. The coordinate system uses + x axis increasing to the right and the +y axis increasing downward. The origin is at the arch crown.

It should be noted that this element applies to linearly elastic arches, and may be used for support displacements and for supports at different elevations, where the defining arch equations are the same on both sides. The effects of uniform temperature changes may be found in Reference 6.

(1) DERIVATION OF BASIC FORCE EQUATIONS

Consider x<0 and node 1

$\sum Fx = 0 \rightarrow +P1*\cos\phi 1+Q1*\sin\phi 1+N*\cos\delta+V*\sin\delta = 0$

$\sum Fy = 0 \rightarrow -P1*\sin\phi 1+Q1*\cos\phi 1-N*\sin\delta+V*\cos\delta = 0$

$\sum M = 0 \rightarrow +M1+P1*\sin\phi 1*(x-x1)-P1*\cos\phi 1*(y1-y)$
$-Q1*\cos\phi 1*(x-x1)-Q1*\sin\phi 1*(y1-y) = 0$

As shown omn the following sketch, $\phi = 2*\pi - \alpha n$ **and** $\delta = 2*\pi - \theta$.

Using trigonometric theorems,

$\sin\Phi n = -\sin\alpha n \qquad \cos\Phi n = \cos\alpha n$

$\sin\delta n = -\sin\theta \qquad \cos\delta n = \cos\theta$

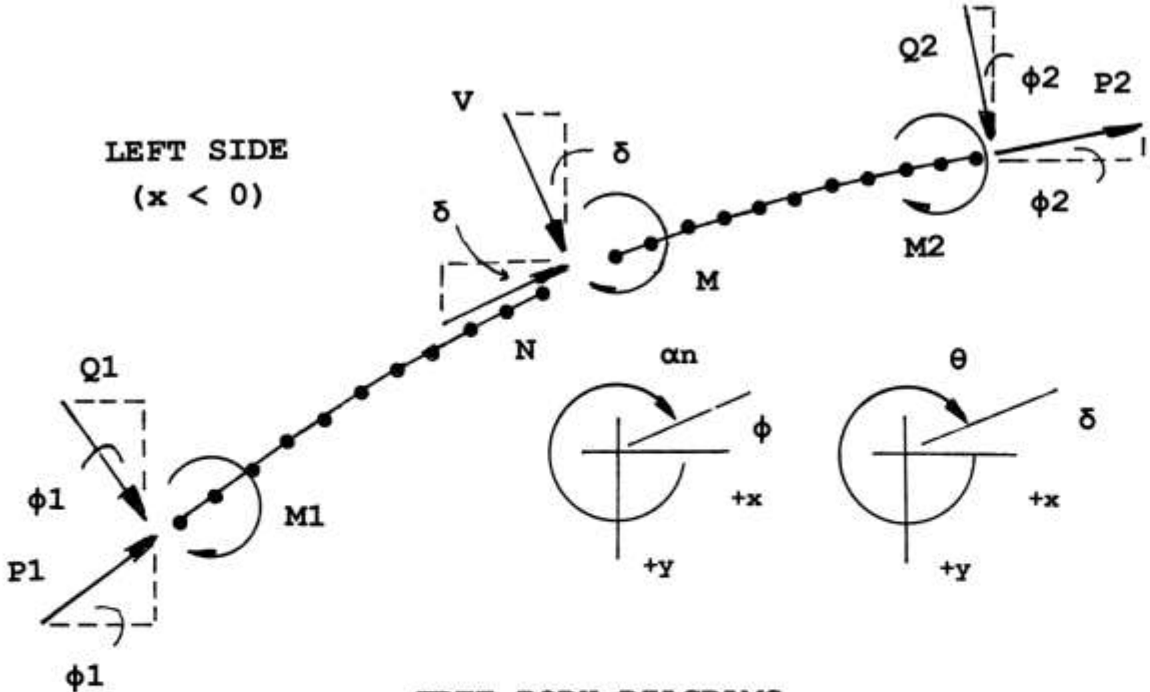
It can be shown that for both nodes and both sides, the same equations (with the exception of node numbers) obtain. Solving the equations,

$N = +Pn*(-\cos\alpha n*\cos\theta-\sin\alpha n*\sin\theta)$
$\quad +Qn*(+\sin\alpha n*\cos\theta-\cos\alpha n*\sin\theta)$

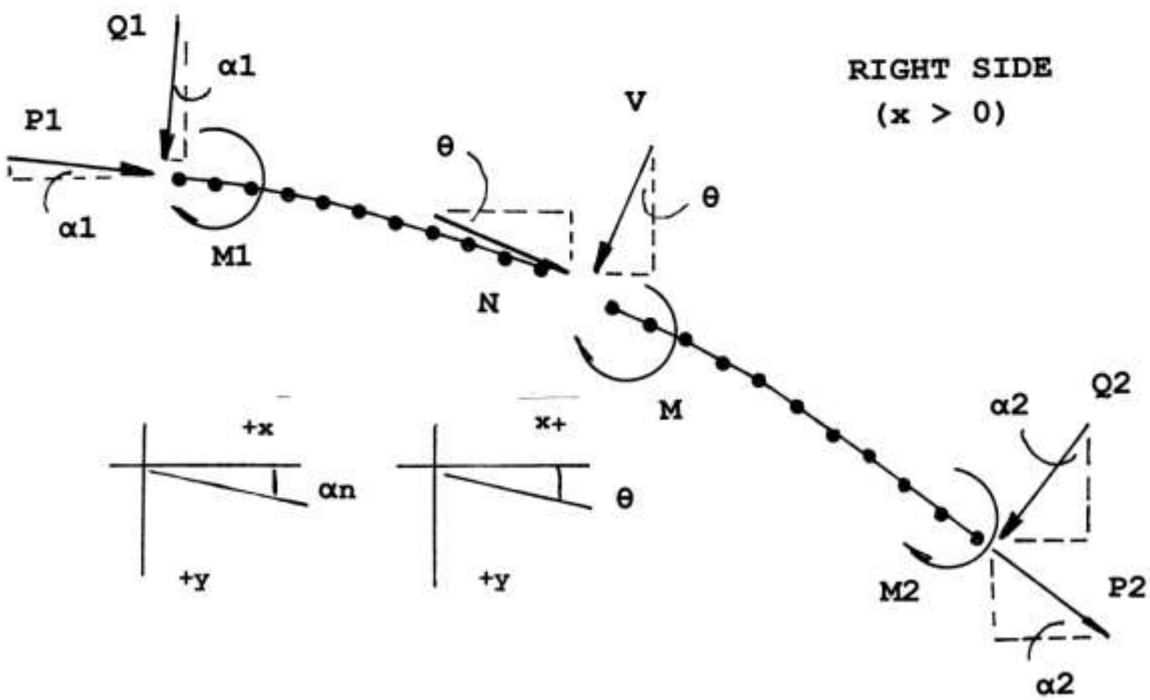
$V = +Pn*(-\sin\alpha n*\cos\theta+\cos\alpha n*\sin\theta)$
$\quad +Qn*(-\cos\alpha n*\cos\theta-\sin\alpha n*\sin\theta)$

$M = -Mn+Pn*(+\sin\alpha n*(x-xn)-\cos\alpha n*(y-yn))$
$\quad +Qn*(+\cos\alpha n*(x-xn)+\sin\alpha n*(y-yn))$

These equations are general and apply to any curve shown with coordinate axes and origin shown.



FREE BODY DIAGRAMS
OF ARC PORTIONS



(2) ADAPTATION OF PARAMETERS TO PARABOLIC ARCH

The equation of a parabola in terms of the coordinate system and origin in (1) above is:

```
4*a*y = x^2, where a = span^2/16*rise
dy/dx = x/2a and dy/dx = tanθ
```

This results in:

```
x  = 2*a*tanθ and
y  = a*(tanθ)^2
also
xn = 2*a*tanαn and
yn = a*(tanαn)^2
```

Now the expressions for N and V remain the same as in part (1), and the equation for M becames

```
M = -Mn +Pn*(+sinαn*(2*a*tanθ-2*a*tanαn)
             -cosαn*(a*(tanθ)^2 -a*(tanαn)^2))
        +Qn*(+cosαn*(2*a*tanθ-2*a*tanαn)
             +sinαn*(a*(tanθ)^2 -a*(tanαn)^2))
```

The radius of gyration of the parabola is:

```
R = (1+(dy/dx)^2)^(1/2)/d^2y/dx^2
```

Substituting into the equations above,

```
R = 2*a*(secθ)^3
and at θ = 0, R0 = 2*a = span^2/8*rise
```

Thus the differential unit of length along the curve, ds, is R*dθ = 2*R0*(secθ)^3, where secθ = 1/cosθ

(3) CALCULATION OF DEFLECTIONS

The method here uses Castigliano's Second Theorem, namely "When a body is elastically deflected in any combination of loads, the deflection at any point and in any direction is equal to the partial derivative of strain energy (computed with all loads acting) with respect to a point located at that point and acting in that direction." This was published as a thesis in 1873. It is similar to, but not identical with, his later method of least work in that both use strain energy terms.

The integrals here are taken over the finite element, not the total structure. The limits of integration are α1 and α2. The integrals shown are those corresponding to the free

body at node 1, with flexibility matrix entries fij shown in red . To calculate equations for node 2, substitute P2, Q2, and M2 for P1, Q1, and M1, respectively. Add '33' to each entry label to find the 3x3 flexibility matrix corresponding to node 2. The term λ adjusts for shape of cross-sectional area in the shear term. Axial, shear, and flexural displacements are u1, v1, and Φ1, respectively. Derivatives shown here are partial.

$$u1 = +(R0/A*E)*\int_{\alpha 1}^{\alpha 2} N*(dN/dP1)*(\sec\theta)^3*d\theta \quad 11,12$$

$$+(\lambda*R0/A*G)*\int_{\alpha 1}^{\alpha 2} V*(dV/dP1)*(\sec\theta)^3*d\theta \quad 11,12$$

$$+(R0/E*I)*\int_{\alpha 1}^{\alpha 2} M*(dM/dP1)*(\sec\theta)^3*d\theta \quad 11,12,13$$

$$v1 = +(R0/A*E)*\int_{\alpha 1}^{\alpha 2} N*(dN/dQ1)*(\sec\theta)^3*d\theta \quad 21,22$$

$$+(\lambda*R0/A*G)*\int_{\alpha 1}^{\alpha 2} V*(dV/dQ1)*(\sec\theta)^3*d\theta \quad 21,22$$

$$+(R0/E*I)*\int_{\alpha 1}^{\alpha 2} M*(dM/dQ1)*(\sec\theta)^3*d\theta \quad 21,22,23$$

$$\Phi 1 = +(R0/A*E)*\int_{\alpha 1}^{\alpha 2} N*(dN/dM1)*(\sec\theta)^3*d\theta \quad = 0$$

$$+(\lambda*R0/A*G)*\int_{\alpha 1}^{\alpha 2} V*(dV/dM1)*(\sec\theta)^3*d\theta = 0$$

$$+(R0/E*I)*\int_{\alpha 1}^{\alpha 2} M*(dM/dM1)*(\sec\theta)^3*d\theta \quad 31,32,33$$

The two Φ1 components are zero as neither N nor V depend on M1. In all cases 21 = 12, 31 = 13 and 32 = 23 as the 3x3 flexibility matrices are symmetric.

The integrals are challenging, as there are eight (8) types to be evaluated, each consisting of one to five terms. The only break is that the integrations need only performed once, as they are identical for both nodes.
The coefficients of the integrals, however, differ from node 1 to node 2. Reference 10 was a big help in evaluating the integrals.
After these calculations, the flexibility corresponding to the 3x3 stiffness matrices, K11 and K22, may be found by inverting these 3x3 flexibility matrices.

(4) CALCULATION OF K21 AND K12

In the equations for N, V, and M in parts (10 and (2), replace N by P1, V by Q1 and, M by M1.
αn → α1 and θ → α2. Thus, letting c = cos s = sin, and t → tan

$$\begin{bmatrix} P2 \\ Q2 \\ M2 \end{bmatrix} = \begin{bmatrix} -c\alpha1*c\alpha2-s\alpha1*s\alpha2 & +s\alpha1*c\alpha2-c\alpha1*s\alpha2 & 0 \\ -s\alpha1*c\alpha2+c\alpha1*s\alpha2 & -c\alpha1*c\alpha2-s\alpha1*s\alpha2 & 0 \\ A+B & C+D & -1 \end{bmatrix} \begin{bmatrix} P1 \\ Q1 \\ M1 \end{bmatrix}$$

A = +2a*sα1*(tα2-tα1)
B = +2a*cα1*(tα2-tα1) 0
C = -a*cα1*((tα2)^2-((tα1)^2))
D = +a*sα1*((tα2)^2-((tα1)^2))

(5) CHECK

After assembling 6x6 stiffness matrix check against References 6 and 9 for a fixed arch when rise = 2 ft, span = 40 ft, with a 10 kip gravity load at the crown.
In this check, the subscript 2 refers to crown properties and the subscript 1 to support quantities.

'Flexure Only' means only flexural strain energy used in calculating deflections.
'Flexure + Axial' means both flexural and axial strain energies used.
'All' means flexural, axial, and shear strain energies used. This only occurs for the program shown here.
Pi and Qi in kip, Mi in kip-in.

	← Flexure Only →		
	Ref. 6	Ref. 9	arc5.c
	-------	-------	-------
P2	46.879	-	46.224
Q2	5.000	-	5.000
M2	225.000	225.424	225.424
P1	46.945	46.894	46.894
Q1	4.290	4.280	4.280
M1	150.000	149.163	149.163

	← Flexure + Axial →		
	Ref . 6	Ref. 9	arc5.c
	-------	-------	-------
P2	44.311	-	44.245
Q2	5.000	-	5.000
M2	245.516	246.154	246.154
P1	44.341	44.366	44.366
Q1	3.787	3.774	3.774
M1	109.969	108.026	108.026

	← All →
	Arc5.c

P2	44.423
Q2	5.000
M2	246.309
P1	44.347
Q1	3.770
M1	107.720

It is seen, that for this case, addition of the shear term changes the quantities very slightly from the results including the flexure and axial terms only.

5. FINITE ELEMENT ANALYSIS

5.1 COORDINATE TRANSFORMATIONS

It is necessary to transform from arc coordinates (P,Q,M and u,v,Δα) to Cartesian coordinates (Fx,Fy,M and Δx,Δy,Δζ) when analyzing finite element assemblies which include straight beam elements as well as arc elements. This involves translation of the 6x6 arc stiffness matrix. Translation back to arc coordinates is needed on a point by point basis when recovering arc force and displacement results from such a program.

The transfer uses the same coordinate system as above, the horizontal axis to the right, the vertical axis downward and positive rotation clockwise.

Consider first the transformation of quantities at a single point from arc coordinates to the Cartesian system Call this system the xo,yo,Φ system.

$$\begin{bmatrix} Fxo \\ Fyo \end{bmatrix} = \begin{bmatrix} +\cos\Phi & -\sin\Phi \\ +\sin\Phi & +\cos\Phi \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} TI \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$$

The same matrix is used for transfer of displacements.

Now generalize this matrix to include both node 1 and node 2. This results in the 6x6 matrix, say 'TI'.

$$TI = \begin{bmatrix} +\cos\alpha1 & -\sin\alpha1 & 0 & 0 & 0 & 0 \\ +\sin\alpha1 & +\cos\alpha1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +\cos\alpha2 & -\sin\alpha2 & 0 \\ 0 & 0 & 0 & +\sin\alpha2 & +\cos\alpha2 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix}$$

α1 = angle at node 1 and α2 the angle at node 2, in arc coordinates. The inverse of this matrix is T, converting Cartesian expressions to arc expressions.

Now let Fxy, Δxy, and Kxy denote parameters in Cartesian coordinates and Fuv, Δuv, and Kuv denote parameters in arc coordinates, for a two-dimensional system.

Dropping the matrix markings, and expressing the above results in the terms immediately above, we have

Fxy = TI*Fuv, Fuv = T*Fxy
Δxy = TI*Δuv, Δuv = T*Δxy
And, for the arc coordinates, the stiffness matrix
Fuv = Kuv*Δuv

Substituting for Fuv and Δuv from above,
T*Fxy = Kuv*T*Δxy
Now multiply each side by TI, the inverse of T.

Fxy = TI*Kuv*T*Δxy which is equivalent to
Fxy = Kxy*Δxy

Where Kxy = TI*Kuv*T

Now the arch may be represented in a Cartesian system which includes straight beam elements.

5.2 SYSTEM ANALYSIS

In this analysis, the global coordinate system is the same as the arc coordinate system, with the crown at the origin, x-axis to the right, and y-axis downward. This has two implications.

First, now transfer of arc finite elements to global coordinates is required.

Secondly, the normal orientation in straight beam finite element programs (axial force vector crossed into perpendicular shear force vector gives CCW moments) has to be changed so that the axial force vector crossed into the perpendicular shear vector gives a CW moment.

Both of these changes are made, as well as providing for pinned, semi-rigid, or fixed ends for the straight beam elements, are present in the developed program globe3.c,

6. STEEL DECK ARCH BRIDGE

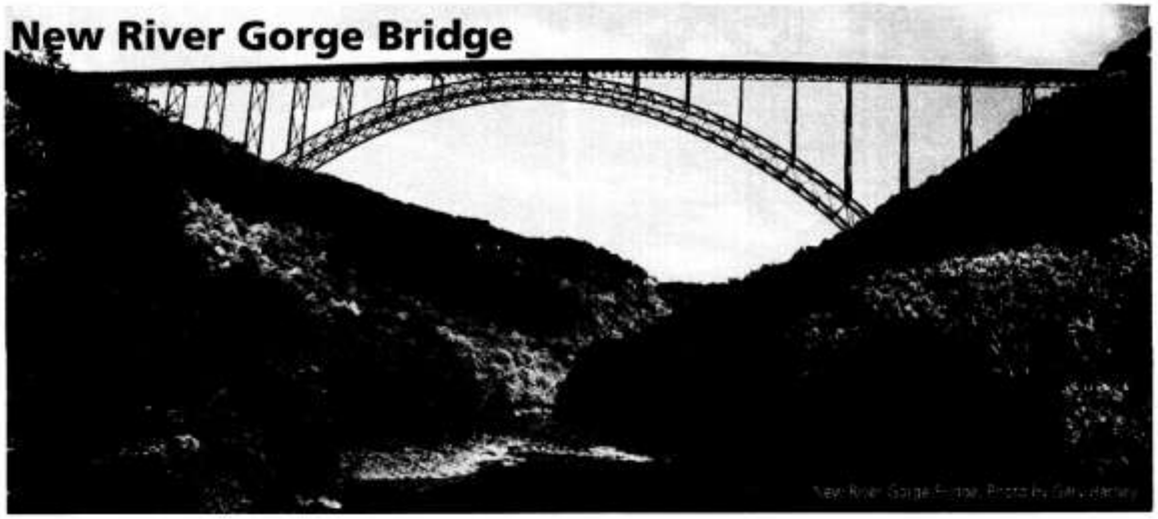
Deck arches, also called true arches, carry the end reactions at the springing directly into the two separate foundations. The arch is entirely below the roadway. They are usually used for crossing deep valleys with steep walls. Rock is generally near the surface of these walls. If, however, the walls have low bearing capacity, the foundation costs greatly increase.

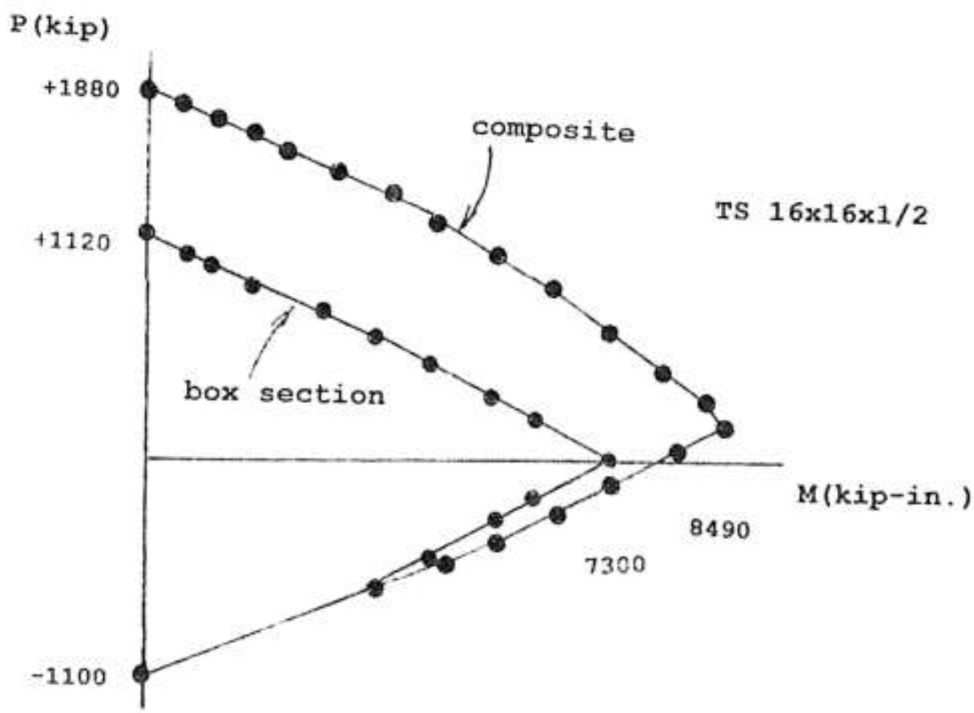
The arch ribs may be either solid or trussed, with trussed ribs suitable for longer spans. The page following shows the longest deck arch in the United States, the New River Gorge Bridge near Fayetteville, West Virginia.

For shorter spans, one choice for arch ribs is a closed box section, which may be filled with concrete for greater compressive capacity. The concrete-filled type is called a composite section.
The steel code allows either a plastic stress distribution method or a strain compatibility method for composite section analysis. The strain compatibility method assumes a maximum concrete stress of .003 inches/inch at the top of the concrete, with stress decreasing linearly to zero downward across the section. The length of this vertical strain is called 'c'and the concrete stress equal to 0.85*β*fc' where fc' is the specified concrete stress at 28 days.

β = 0.85, fc' < 4000psi
= 0.85-0.05*(fc'-4000)/1000, 4000psi <= fc' <= 8000psi
= 0.65, fc' > 8000 psi

The page following also shows both the composite and box only force versus moment diagrams for a large structural square tubing, using the strain compatibility method. The values plotted are those of nominal capacity, which should be used with the strength design method.
The strength design method also includes capacity reduction factors and multiplication of the service loads.


New River Gorge Bridge


P(kip)
+1880
+1120
-1100
composite
TS 16x16x1/2
box section
M(kip-in.)
8490
7300

Preliminary analysis and/or design of a steel deck arch bridge considers buckling in the plane of the arch, axial and flexural stresses, moment magnification,, moments due to non-uniform dead load, and type of connections of spandrel columns to the arch rib.

Buckling in Plane of the Arch

As described in Reference 11, the buckling length for a two-hinged arch is one half of the total arc length = L.

$$L = \int_{-x0}^{+x0} (1+(dy/dx)2)^{(1/2)}dx, \text{ where } dy/dx = x/2a$$

Using the integral tables in Reference__,

L = x*(x^2+4a^2)^(1/2) + 2*a^2*ln(x+(x^2+4*a^2)^(1/2)) evaluated from –x0 to +x0.

Moments due to Non-uniform Dead Load

The majority of the dead load is applied as point loads at the spandrel column locations. These moments are determined using the system finite element approach described in Section 5. above.

Stresses and Moment Magnification

Axial and flexural stresses are determined when cross-sectional properties of arch rib, spandrel columns, and roadway depth are known. The moment in the arch is increased by the deflection of the arch from the center line multiplied by the axial force, due to both dead and live load.

The magnification Af = 1/(1-S.F.*T/A*Fe) where
T = arch rib thrust
S.F. = safety factor
A = arch rib cross-sectional area
Fe = ∏^2*E/(KL/r)^2
E = modulus of elasticity

K depends upon the arch type and rise/span ratio as:

Rise/Span	3-Hinged	2-Hinged	Fixed
0.1-0.2	1.16	1.04	0.70
0.2-0.3	1.13	1.10	0.70
0.3-0.4	1.16	1.16	0.72

L = length of arc rib/2
r = (I/A)^(1/2)
I = strong axis moment of inertia
S = I/(depth/2) for symmetric cross-sections

Types of Connections

The moments induced in the arch ribs and the spandrel columns are a strong function of the degree of restraint to moment of the connections. AISC defines the restraints as being of three types, namely simple,partially restrained (PR), and fully restrained (FR). These restraints are also known as pinned, semi-rigid, and rigid. The connected are defined as

simple : ks < = 2*E*I/l
PR : 2*E*I/l < ks < 20*E*I/l
FR : 20*E*I/l <= ks
ks = connection stiffness, usually in kip-in./radian
l = column length

Example of Two-Hinged Steel Deck Arch Bridge
****

Span = 300 ft
Rise = 40 ft
Arch Rib = box section
flanges = pl 2″ x 30″
webs = pl 1″ x 56″
ASTM F1554, Grade 36
Arch Rib + Bracing DL = 1.5 kip/ft
Deck DL = 3.0 kip/ft
Deck LL = 0.640 kip/ft

Arch Rib Section Properties

I = 137547 in.^4
S = 4584.8 in.^3
A = 260.00 in.^2
r = 23.001 in.

Buckling in the Plane of the Arch

Rise/Span = .1333 so that K = 1.04
The arc rib length/2 calculates as 1882.02″
Fa = allowable axial stress
Fa = Fy/(5/3+3*(Kl/r)/8*Cc-((Kl/r)/8*Cc)^3)
Fa = 19.136 ksi

Moments Due to Total Dead Load

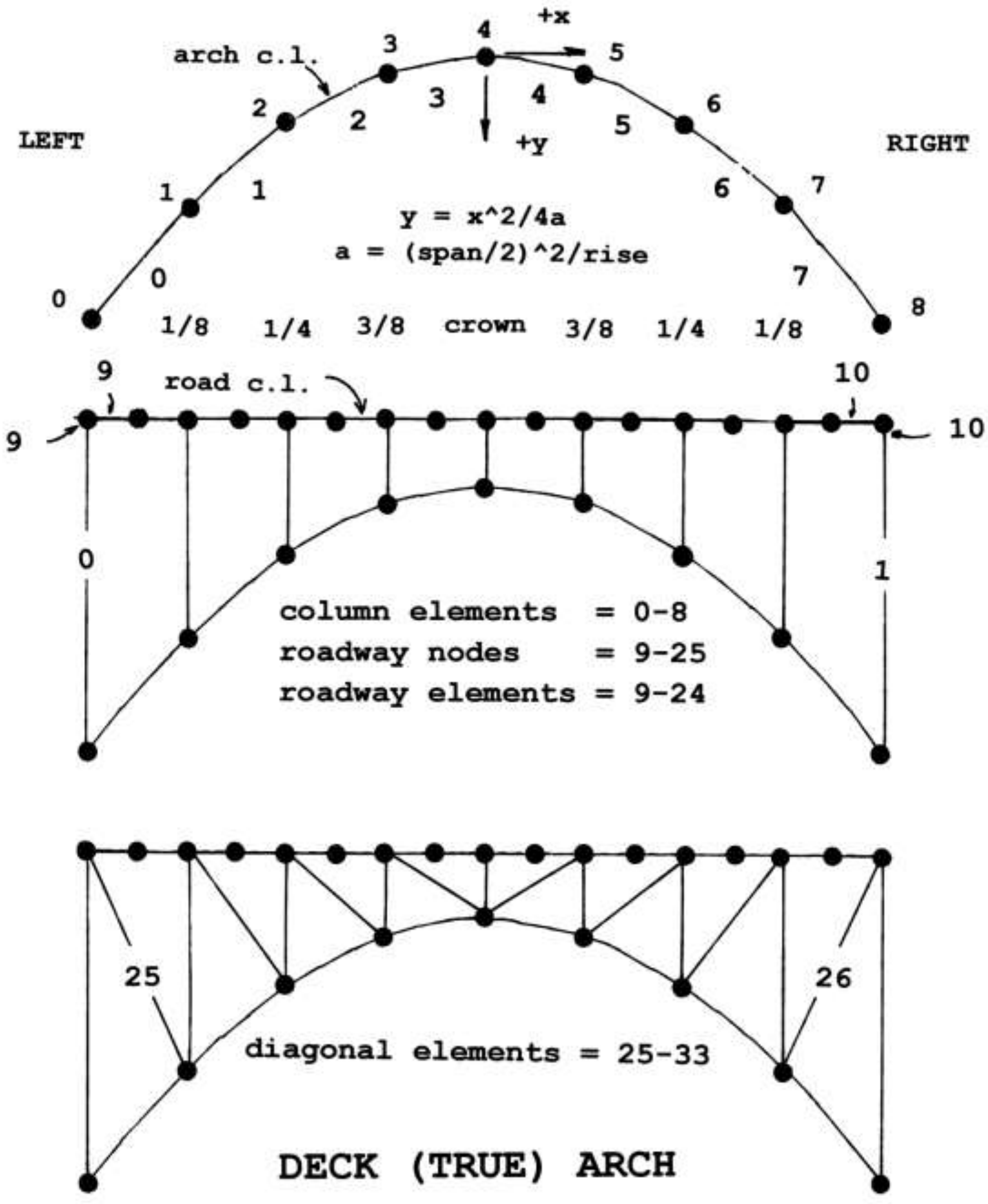
The diagrams on the next page show arch alone, arch with columns and deck, and arch with columns, deck and diagonals.
First consider the spandrel columns only, maximum values shown.
Δ = arch deflection N, V, and M are arch values.
Connections refer to spandrel column connections.

Conn.	Δ(in.)	N(kip)	V(kip)	M(k″)
Simple	1.713	1494	116.1	19582
Fixed	1.471	1507	109.4	17147

Now consider the spandrel columns with diagonals.
Connections refer to both columns and diagonals

Conn.	Δ(in.)	N(kip)	V(kip)	M(k″)
Simple	0.689	1020	119.8	21020
Fixed	0.804	1045	104.7	20670

In this example the use of diagonals lowers the maximum axial force, but it achieved at the cost of extra material, more complicated connections, and increased moment.
It should also be noted that simple connections eliminate column moments.



DECK (TRUE) ARCH

7. TIED ARCHES

Tied arches balance the arch horizontal reactions by a tension tie between the ends of the arch. Tied arches are generally the through type, although the bridge may be constructed with the deck midway between the arch crown and the springing.

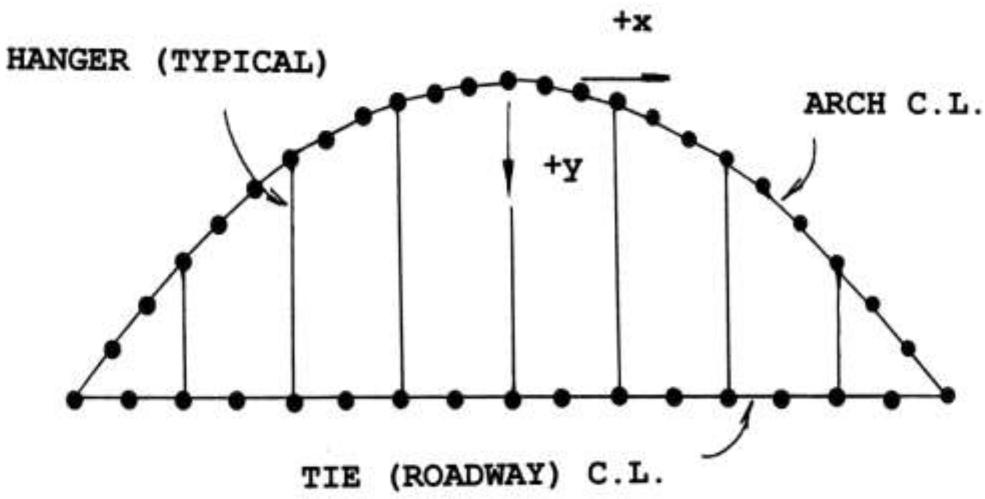
Tied arches are generally used where deep foundations are required. In the tied arch, only vertical loads need be transmitted to earth by the foundations.

Load is transmitted from the roadway to the arch by hangers in tension. Hangers in the United States have traditionally been of the two or four bridge strand or bridge rope. Traditional hangers have been vertical. Recent designs have examined network hangers. These systems connect arch and deck at non-vertical angles.

Moment magnification is not used for tied arches. At any point, the tie and the arch deflect approximately the same amount. Therefore the moment arm between the tension in the tie and the horizontal component of thrust remains the same.

Unlike the deck arch, temperature change has little effect on the arch forces if the arch and tie remain at essentially the same temperature.
The division of live moment between he arch rib and tie girder depends upon their respective moments of inertia, As shown in the following example.

Example of Two-Hinged Steel Tied Arch Bridge
****
Use the same properties as the deck arch example.
300 ft span, 40 ft rise
Arch Rib + Bracing DL = 1.5 kip/ft
Deck DL = 3.0 kip/ft
Deck LL (half span) = 0.640 kip/ft
Arch cross section: A = 120 in.^2, I = 137347 in.^4
Deck area and moment of inertia varies as shown.



Three cases are examined. The ratio of tie area and inertia to the arch area and inertia is varied from 0.5 to 2.0 with the results shown. Hangers are assumed to be four (4) inches in diameter.

Ratio →	0.5	1.0	2.0
Δ arch, hor. (in.)	1.300	0.651	0.394
Δ arch, vert. (in.)	3.204	2.212	1.052
N, max. (kip)	1473	1472	1464
V, max. (kip)	98.4	98.7	99.0
M, max. (kip-in.)	20244	16133	13329
Tension, deck (kip)	1324	1327	1324
M, max, deck (k″)	11312	14471	19074
Δ deck, max. (in.)	3.367	2.371	1.807

It is seen that the major effect of increasing ratio is decrease of arc moment and increase of deck (tie) moment. In this example, all hangers are in tension, but in general the hangers must be pre-tensioned to avoid loss of tension for some loads.

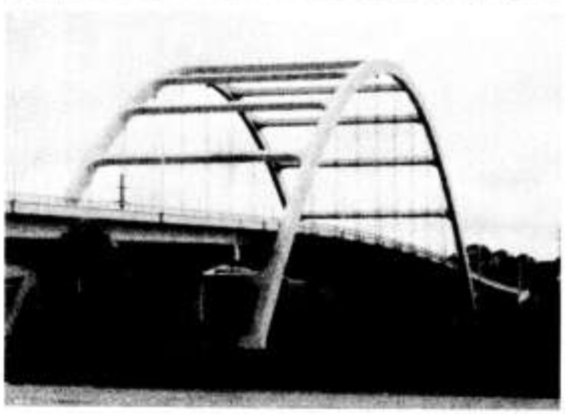
A tied-arch bridge is really a simply supported beam, having a pinned connection at one end and only vertical support at the other end. It is free to move horizontally with load and/or temperature.

The following tied-arch bridge illustrations are taken from "Salem River Crossings, Bridge Type Discussion", Task Force Meeting, January 16, 2006.

top left. Blennerhassett – West Virginia

bottom left. Gateway Boulevard - Nashville *(Partial Through Arch)*

below. Page Avenue - Missouri



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APPENDIX 1

H0.c is a program to find the redundant horizontal reaction for a two-hinged arch.

```
                    START
                      |
         DECLARATION OF VARIABLES
                      |
        INPUT ARCH CHARACTERISTICS
                      |
        SOLVE FOR FLEXURAL STRAIN
           ENERGY DENOMINATOR
                      |
        SOLVE FOR FLEXURAL STRAIN
            ENERGY NUMERATOR
                      |
     AS ABOVE FOR AXIAL STRAIN ENERGY
                      |
     AS ABOVE FOR SHEAR STRAIN ENERGY
                      |
        SOLVE FOR THE REDUNDANT H0
                      |
    PRINT M, V, N FOR SELECTED POINTS
                      |
                   FINISH
```

```
/*****************************************************************
 *                                                               *
 *                                                               *
 * H0.c : Castigliano's least work theorem :                     *
 *      Two-Hinged Arch : Solve for horizontal reaction          *
 *        and plot N,V, and M                                    *
 *        See text for explanations of integrals                 *
 *      div = number of plotted points                           *
 *      mode = 1 for flexural strain energy                      *
 *             2 for 1 plus axial strain energy                  *
 *             3 for 2 plus shear strain energy                  *
 *      inputs in consistent units, eg., kip, inches             *
 *                                                               *
 *****************************************************************/

#include<math.h>
#include<stdio.h>
#include<stdlib.h>
int main(void)
{
            /* DECLARATIONS */
```

```
int div,mode;
double A,E,eta,F1,G,I,V0,x0,x1,y0;
double a,alf,arg0,arg1,H0,H1,M,N,R0,theta,theta0,theta1,V,x,y;
double II1,II2,II3,II4,II5,II6,II7,II8,II9,II10,II11,II12,II13;
double num,den;
double pi   =     3.141592653589793;
FILE *inn;
FILE *out;

      /* INPUTS AND PARAMETERS */

inn   =     fopen("H0.in","r");
out   =     fopen("H0.out","w+");
fscanf(inn,"%i %i %lf%lf %lf %lf %lf %lf %lf %lf %lf %lf",
          &div,&mode,&A,&E,&eta,&F1,&G,&I,&V0,&x0,&x1,&y0);
fclose(inn);

      /* CALCULATE ARCH PARAMETERS */

a     =     x0*x0/(4.0*y0);
alf   =     2.0*a;
arg0  =     alf*alf+x0*x0;
arg1  =     alf*alf+x1*x1;
R0    =     2.0*a;
theta0      =     atan2(x0,(2.0*a));
theta1      =     atan2(x1,(2.0*a));

      /* H0 (flexural strain energies) INTEGRALS */

II1   =     +x0*sqrt(arg0)
            +(alf*alf/2.0)*(log(+x0+sqrt(arg0))
            -log(-x0+sqrt(arg0)));
II1   =     +(y0*y0/alf)*II1;
II1   =     +II1/(E*I);

II2   =     +(x0/2.0)*arg0*sqrt(arg0)
            -(alf*alf/4.0)*x0*sqrt(arg0)
            -(alf*alf*alf*alf/8.0)*(log(x0+sqrt(arg0))
            -log(-x0+sqrt(arg0)));
II2   =     -(y0/(alf*alf))*II2;
II2   =     +II2/(E*I);

II3   =     +2.0*x0*x0*x0*arg0*sqrt(arg0)/3.0
            -x0*arg0*arg0*sqrt(arg0)/3.0
            +alf*alf*x0*arg0*sqrt(arg0)/12.0
            +alf*alf*alf*alf*x0*sqrt(arg0)/8.0
            +alf*alf*alf*alf*alf*alf*(log(x0+sqrt(arg0))
            -log(-x0+sqrt(arg0)))/16.0;
II3   =     +II3/(4.0*alf*alf*alf);
II3   =     +II3/(E*I);

      /* V0 (flexural strain energies) INTEGRALS */
```

```
II4   =    +x0*arg0*sqrt(arg0)/2.0-(alf*alf*x0/4.0)*sqrt(arg0)
           -(alf*alf*alf*alf/8.0)*(log(+x0+sqrt(arg0))
           -log(-x0+sqrt(arg0)));
II4   =    -(V0*x0/(2.0*alf*alf))*II4;
II4   =    +II4/(E*I);

II5   =    +x0*sqrt(arg0)+(alf*alf/2.0)*(log(+x0+sqrt(arg0))
           -log(-x0+sqrt(arg0)));
II5   =    +(V0*x0*y0/alf)*II5;
II5   =    +II5/(E*I);

      /* F1 (flexural strain energies) INTEGRALS */

II6   =    +arg0*arg0*sqrt(arg0)/5.0-alf*alf*arg0*sqrt(arg0)/3.0
           -arg1*arg1*sqrt(arg1)/5.0+alf*alf*arg1*sqrt(arg1)/3.0;
II6   =    +F1*II6/(2.0*alf*alf);
II6   =    +II6/(E*I);

II7   =    +arg0*sqrt(arg0)/3.0
           -arg1*sqrt(arg1)/3.0;
II7   =    -F1*y0*II7/alf;
II7   =    +II7/(E*I);

II8   =    +x0*arg0*sqrt(arg0)/4.0-alf*alf*x0*sqrt(arg0)/8.0
           -(alf*alf*alf*alf/8.0)*log(x0+sqrt(arg0))
           -x1*arg1*sqrt(arg1)/4.0+alf*alf*x1*sqrt(arg1)/8.0
           +(alf*alf*alf*alf/8.0)*log(x1+sqrt(arg1));
II8   =    -F1*x1*II8/(2.0*alf*alf);
II8   =    +II8/(E*I);

II9   =    +x0*sqrt(arg0)/2.0+(alf*alf/2.0)*log(x0+sqrt(arg0))
           -x1*sqrt(arg1)/2.0-(alf*alf/2.0)*log(x1+sqrt(arg1));
II9   =    +F1*x1*y0*II9/alf;
II9   =    +II9/(E*I);

      /* H0 (axial strain energy) INTEGRALS */

II10  =    +log(tan(+theta0/2.0+pi/4.0))
           -log(tan(-theta0/2.0+pi/4.0));
II10  =    +II10*R0/(E*A);

      /* F1 (axial strain energy) INTEGRALS */

II11  =    +1.0/cos(theta0)
           -1.0/cos(theta1);
II11  =    -II11*R0*F1/(E*A);

      /* H0 (shear strain energy) INTEGRALS */

II12  =    +tan(theta0)*tan(theta0)*sin(theta0)+sin(theta0)
           -(1.0/2.0)*log(tan(+theta0/2.0+pi/4.0))
           +(1.0/2.0)*log(tan(-theta0/2.0+pi/4.0));
II12  =    +II12*eta*R0/(G*A);
```

```
          /* F1(shear strain energy) INTEGRALS */

     II13  =     +1.0/cos(theta0)
                 -1.0/cos(theta1);
     II13  =     +II13*eta*R0*F1/(G*A);

          /* CALCULATE AND PRINT H0exact */

     switch(mode)
     {
          case 1:
               num   =     +II4+II5+II6+II7+II8+II9;
               den   =     +II1+II2+II3;
               H0    =     num/den;
               break;
          case 2:
               num   =     +II4+II5+II6+II7+II8+II9+II11;
               den   =     +II1+II2+II3+II10;
               H0    =     num/den;
               break;
          case 3:
               num   =     +II4+II5+II6+II7+II8+II9+II11+II13;
               den   =     +II1+II2+II3+II10+II12;
               H0    =     num/den;
               break;
          default: ;
     }

     fprintf(out,"H0exact    = ");fprintf(out,"%21.9f\n\n",H0);

          /* PLOT x,N,V.M */

     for(x=-x0;x<=+x0;x+=2.0*x0/div)
     {
          theta =     atan2(x,(2.0*a));
          y     =     x*x/(4.0*a);
          if(x<x1)
          {
               N     =     -H0*cos(theta)+V0*sin(theta);
               V     =     +V0*cos(theta)+H0*sin(theta);
               M     =     +H0*(y0-y)-V0*(x+x0);
          }
          else
          {
               N     =     -H0*cos(theta)+(V0-F1)*sin(theta);
               V     =     +(V0-F1)*cos(theta)+H0*sin(theta);
               M     =     +H0*(y0-y)-V0*(x+x0)+F1*(x-x1);
          }
          fprintf(out,"%6.1f %16.6e %16.6e %16.6e\n",x,N,V,M);
     }

     fclose(out);
     return 0;
}
```

APPENDIX II

The main function of arc5.c, a program to generate the stiffness matrices, both in uv-coordinates and xy-coordinates is shown in this appendix.

The following table relates the subroutines in arc5.c to their function. The element numbers differ by one less than those in the text because arrays in C start with label '0', not '1'. The text nomenclature, starting with '1' for arrays, is common in the literature. It should also be noted that literature examples seldom include strain energy shear terms.

SUBROUTINE	EVALUATES
integrals	**required integrals, from first to second node**
matrix	**required coefficients for each integral**
findf	**combines the above two subroutines to find the 3x3 flexibility matrices. fm, fp, and fq that represent the flexure, axial, and shear terms, respectively**
crossover	**find K21 given K11**
transfer	**convert stiffness matrix in terms of du, dv, and dθ to one in terms of rectangular coordinates dx, dy, and dθ**

```
/*   ARC5.c
     calculates 6x6 stiffness matrix for member with:
     (1)  centerline = parabola defined by y = x^2/4a where
          a = span^2/4*rise
     (2)  abuments at same elevation - if abutments at different
           elevations, redefine the axis from x = level, y = down, to
           x = bridge line. y axis perpendicular to x, and loads
           adjusted to new axes.
     (3)  member is prismatic (constant cross-section)
     (4)  x0,y0,x1,y1 nodes near and far (0 and 1) coordinates
     (5)  theta0, theta1 nodes 0 and 1 angles
     (6)  A      = cross-section area, in.^2
     (7)  E      = flexural and axial modulus of elasticity, ksi
     (8)  eta     = 1.2, correction factor for rectangular section
     (9)  G = shear modulus of elasticity
     (10) I      = strong axis moment of inertia, in.^4
     (11) R0     = radius of curvature at crown
     (12) KUV    = 6x6 stiffness matrix, in terms of a, theta
     (13) KXY    = 6x6 stiffness matrix, in therms of x,y
```



```
     (14) KA    = 3x3 stiffness matrix, upper left, in terms of a,
                  theta
     (15) KB,KC = crossover 3x3 stiffness matrices
     (16) KD    = 3x3 stiffness matrix, lower right, in terms of a,
                  theta
     (17) ff    = working 3x3 stiffness matrix
     (18) loc   == 0 -> 00 3x3 matrix loc!= 0 -> 11 3x3 matrix
     (19) zed   = number of flexibility terms used as :
                   1 = flexure only, 2 = 1 + axial,* 3 = 2 + shear */
#include<math.h>
#include<stdio.h>
#include<stdlib.h>
#include"hdrinv3.h"

int main(void)
{
            /* INPUT DATA */

      FILE *inn;
      FILE *out;
      inn   =     fopen("arc5.in","r");
      out   =     fopen("arc5.out","w+");
      fscanf(inn,"%lf %lf %lf %lf %lf %lf %lf %lf %lf %i",
                  &AA,&EE,&eta,&GG,&II,&rise,&span,&X0,&X1,&zed);
      fclose(inn);

            /*  USE PARAMETRIC EQUATIONS TO FIND aa,R00,theta0,theta1 */

      aa    =     span*span/(16.0*rise);
      R00   =     2.0*aa;
      theta0      =     atan2((X0/(2.0*aa)),1.0);
      theta1      =     atan2((X1/(2.0*aa)),1.0);

            /* ZERO OUT ARRAYS */

      for(ii=0;ii<=13;ii++)
      {
            rr[ii]      =     0.0;
      }

      for(ii=0;ii<=7;ii++)
      {
            tt[ii]      =     0.0;
      }

      for(ii=0;ii<=2;ii++)
      {
            for(jj=0;jj<=2;jj++)
            {
                  ff[ii][jj]  =     0.0;
                  KA[ii][jj]  =     0.0;
                  KB[ii][jj]  =     0.0;
                  KC[ii][jj]  =     0.0;
                  KD[ii][jj]  =     0.0;
```

```
        }
    }

    for(ii=0;ii<=5;ii++)
    {
        for(jj=0;jj<=5;jj++)
        {
            KUV[ii][jj] =     0.0;
            KXY[ii][jj] =     0.0;
        }
    }

        /* FIND INTEGRATION MATRIX */

    integrals(theta0,theta1,tt);

        /* FIND UPPER LEFT 3x3 STIFFNESS MATRIX */

    loc   =     0;

    matrix(loc,aa,theta0,theta1,rr);

    findf(zed,eta,AA,EE,GG,II,R00,tt,rr,ff);

    invert3x(ff);

    for(ii=0;ii<=2;ii++)
    {
        for(jj=0;jj<=2;jj++)
        {
            KA[ii][jj]  =     ff[ii][jj];
        }
    }

        /* FIND CROSSOVER MARTICES, UPPER RIGHT AND LOWER LEFT */

    crossover(aa,theta0,theta1,KA,KC);

    for(ii=0;ii<=2;ii++)
    {
        for(jj=0;jj<=2;jj++)
        {
            KB[jj][ii]  =     KC[ii][jj];
        }
    }

        /* FIND LOWER RIGHT 3x3 MATRIX */

    loc   =     1;

    matrix(loc,aa,theta0,theta1,rr);

    findf(zed,eta,AA,EE,GG,II,R00,tt,rr,ff);
```

```
	invert3x(ff);

	for(ii=0;ii<=2;ii++)
	{
		for(jj=0;jj<=2;jj++)
		{
			KD[ii][jj]  =     ff[ii][jj];
		}
	}

		/* FIND KUV AND PRINT */

	for(ii=0;ii<=2;ii++)
	{

		for(jj=0;jj<=2;jj++)
		{
			KUV[ii][jj] =     KA[ii][jj];
			KUV[ii][jj+3]     =     KB[ii][jj];
			KUV[ii+3][jj]     =     KC[ii][jj];
			KUV[ii+3][jj+3]   =     KD[ii][jj];
		}
	}

	fprintf(out,"KUV[6][6]\n");fprintf(out,"---------\n\n");

	for(ii=0;ii<=5;ii++)
	{
		for(jj=0;jj<=5;jj++)
		{
			fprintf(out,"%18.9e",KUV[ii][jj]);
		}
		fprintf(out,"\n");
	}
	fprintf(out,"\n\n\n");

		/* FIND KXY AND PRINT */

	transfer(theta0,theta1,KUV,KXY);

	fprintf(out,"KXY[6][6]\n");fprintf(out,"---------\n\n");

	for(ii=0;ii<=5;ii++)
	{
		for(jj=0;jj<=5;jj++)
		{
			fprintf(out,"%18.9e",KXY[ii][jj]);
		}
		fprintf(out,"\n");
	}
	fclose(out);
	return 0;
}
```